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# APPLIED MECHANICS

BY

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## PREFACE

This book is intended for use with students of engineering. It is the outgrowth of the course in mechanics which for several years has been given to the sophomore engineering students in this institution.

While it is supposed that students using the book will have had a course in college physics, it is not essential that such be the case. An attempt has been made to develop the course logically from fundamental principles, so that a student who has a fair working knowledge of mathematics through the integral calculus can successfully carry the course.

The material dealt with is to a large extent of engineering interest and the main purpose of its study is to develop in the student an ability to reason logically concerning the behavior of the materials and forces with which he will deal as an engineer.

A large number of problems is given—more than can be solved by the student in the time ordinarily allotted to such a course. The answers are given, for I believe that the student gains most from such a study when he can check his results.

My thanks are due to Professor M. M. Frocht, who read portions of the manuscript and proof, and to Mr. J. J. Stoker, who has verified much of the numerical work in the text and has checked most of the answers to the problems.

N. C. RIGGS

CARNEGIE INSTITUTE OF TECHNOLOGY  
PITTSBURGH, PA.  
May, 1930



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## APPLIED MECHANICS





# APPLIED MECHANICS

## CHAPTER I

### FORCES IN ONE PLANE

**1. Introduction.** The student taking up this work is supposed to have had a course in College Physics and to be familiar with the principles of mechanics usually included in such a course. In order to make a connected treatment of the subject, however, and to provide a review, the elementary principles will be briefly stated and proofs of the important theorems will be given in the development of the course.

**2. Vectors.** In the study of mechanics two classes of quantities are dealt with: those which have magnitude only, and those which depend upon magnitude and direction. In the former class may be mentioned time, volume, mass, and density. Quantities of this class are called *scalars*. In dealing with quantities involving magnitude and direction, it is often convenient to make use of vectors.

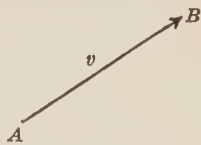


FIG. 1

A *vector* is a quantity which may be represented by a directed segment of a straight line and when combined with another quantity of the same kind, obeys the law of addition stated in the next article. Thus Fig. 1 represents a vector. The arrow head represents direction from A, the *initial point*, to B, the *terminal point*. The distance from A to B, to a definite scale, determines the magnitude of the vector. If, as sometimes is the case, the point from which the vector is laid off is immaterial, the vector is called a *free* vector. Otherwise it is a *restricted* vector. For convenience of discussion the segment which represents a vector will itself be called the vector. The vector is read by naming the initial point and the terminal point in succession,

thus,  $AB$ . Or the vector may be read as a single letter, as  $v$ , placed alongside the vector. Reversing the direction of a vector exchanges initial and terminal points. Reversal of direction is also indicated by prefixing a minus sign to the symbol. Thus  $-AB = BA$ .

**3. Addition of Vectors.** Vector addition of free vectors is defined as follows.

*A vector  $b$  is added to a vector  $a$  by placing the vector  $b$  with its initial point upon the terminal point of the vector  $a$ ; the vector extending from the initial point of  $a$  to the terminal point of  $b$  is then the vector sum of  $a$  and  $b$ .*

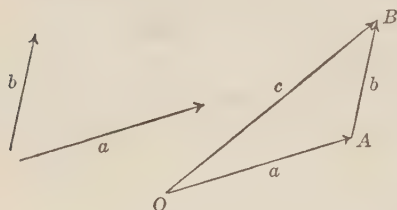


FIG. 2

Thus, in Fig. 2, where  $OA$  is equal to the vector  $a$  and  $AB$  is equal to the vector  $b$ , the vector  $OB$ , or  $c$ , is the vector sum of  $a$  and  $b$ . It is evident from the definition that the result of adding

$a$  to  $b$  is the same as the result of adding  $b$  to  $a$ .

The sum of three or more vectors is obtained by a successive application of the process of addition of two vectors.

Thus in Fig. 3,  $OD$  is the vector sum of the vectors  $a$ ,  $b$ ,  $c$ , and  $d$ . This result is obtained by first taking the sum of  $a$  and  $b$ , obtaining  $OB$ , adding  $c$  to  $OB$  to obtain  $OC$ , and adding  $d$  to  $OC$  to obtain  $OD$ . The vectors  $OB$  and  $OC$  may be left out of the construction and the rule of addition may be stated as follows.

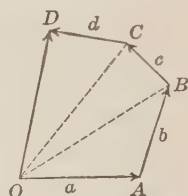
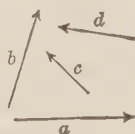


FIG. 3

*To sum any number of free vectors lay off the vectors in succession, placing the initial point of each vector upon the terminal point of the preceding; the vector extending from the initial point of the first vector to the terminal point of the last is their vector sum.*

Here again it is easily seen that the same result is obtained by

taking the vectors in any order. The method applies whether or not the vectors are all in the same plane. The method is also applicable to a set of restricted vectors with a common initial point.

**4. Subtraction of Vectors.** From the ordinary meaning of subtraction it follows that if  $b$  added to  $a$  equals  $c$ , then  $b$  subtracted from  $c$  equals  $a$ , or  $a$  subtracted from  $c$  equals  $b$ . Thus, in Fig. 2, the vector  $AB$  is equal to the vector  $OB$  minus the vector  $OA$ , and the vector  $OA$  is equal to the vector  $OB$  minus the vector  $AB$ . We may then state the rule for the subtraction of vectors in either of two ways: (1) *The vector difference  $a - b$  is the vector extending from the initial point of  $a$  to the initial point of  $b$  when the vectors are placed with their terminal points coinciding*, or (2) *The vector difference  $a - b$  is the vector extending from the terminal of  $b$  to the terminal of  $a$  when the vectors are placed with their initial points coinciding*. Thus, in Fig. 4, either  $OB$  in (1) or  $BA$  in (2) represents the vector difference  $a - b$ .

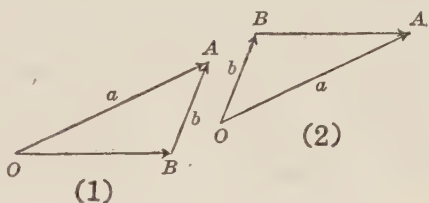


FIG. 4

**5. Components of a Vector.** Vectors  $a, b, c, \dots$  which added produce a vector  $r$  are called the **components** of  $r$ , and  $r$  is called the **resultant** of  $a, b, c, \dots$ . When  $r$  is replaced by its components it is said to be “resolved into its components.” Evidently a set of

components in an infinite number of ways.

If a vector is resolved into two components at right angles to each other, these are called **rectangular components**

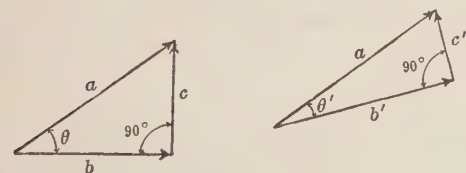


FIG. 5

of the vector in the plane. Thus, in Fig. 5, either  $b$  and  $c$  or  $b'$  and  $c'$  are rectangular components of the vector  $a$ . It is evident

that the values of the components depend upon the directions chosen for the components as well as upon the vector  $a$ .

From the figure it is seen that the numerical values of the rectangular components are given by the equations

$$b = a \cos \theta, \quad c = a \sin \theta,$$

in which  $a$ ,  $b$ , and  $c$  are taken to mean the numerical values, or magnitudes, of the vectors.

**6. Computation of the Resultant of Coplanar Vectors.** Having given any number of vectors in one plane, the magnitude and direction of their resultant is found by choosing two rectangular axes and resolving the vectors into components along these axes, making the usual convention of signs. Thus, in Fig. 6, the vectors  $a_1, a_2, a_3, \dots$  are resolved into components along two rectangular axes  $OX$  and  $OY$ , directions to the right and upward being counted as positive and those to the left and downward negative. The component vectors along the axes are then added algebraically and these resultant components are then combined into a single

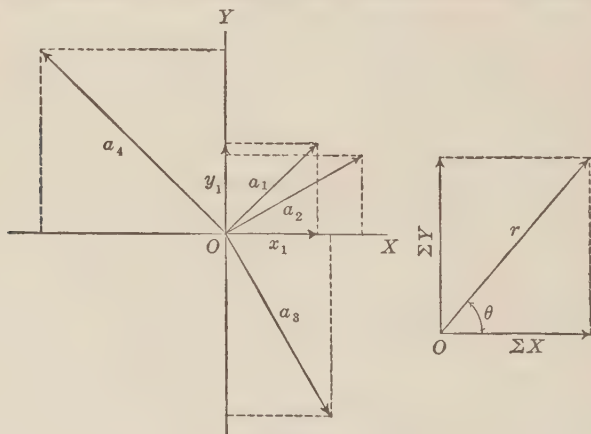


FIG. 6

resultant. If the sum of the components along  $OX$  and  $OY$  are represented respectively by  $\Sigma X$  and  $\Sigma Y$ , and if  $r$  is the magnitude of the resultant vector and its direction makes an angle  $\theta$  with the positive direction of the  $x$  axis, then

$$r^2 = (\Sigma X)^2 + (\Sigma Y)^2, \quad \tan \theta = (\Sigma Y)/(\Sigma X).$$

As an illustration, suppose the numerical values of the vectors shown in Fig. 6 are

$$a_1 = 5, \quad a_2 = 6, \quad a_3 = 8, \quad a_4 = 10,$$

and that the angles which their directions make with the positive direction of the  $x$  axis are respectively,

$$45^\circ, \quad 30^\circ, \quad -60^\circ, \quad 135^\circ.$$

Then

$$\Sigma X = 5 \cos 45^\circ + 6 \cos 30^\circ + 8 \cos 60^\circ - 10 \cos 45^\circ = 5.66,$$

$$\Sigma Y = 5 \sin 45^\circ + 6 \sin 30^\circ - 8 \sin 60^\circ + 10 \sin 45^\circ = 6.68.$$

Hence

$$r = [(5.66)^2 + (6.68)^2]^{1/2} = 8.76, \quad \theta = \tan^{-1} (6.68/5.66) = 49^\circ 43'.$$

**7. Computation of the Resultant of Vectors Not in the Same Plane.** When the vectors are not in the same plane, a set of rectangular axes in space may be assumed and the vectors resolved into components along the axes. Thus, in Fig. 7, the

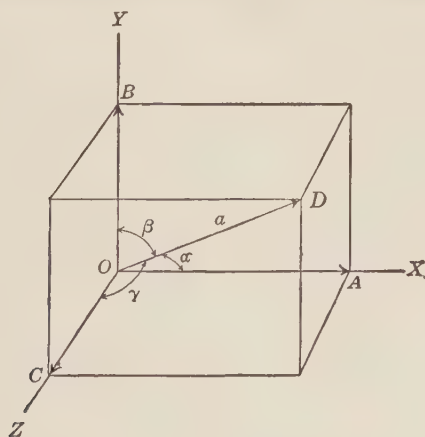


FIG. 7

vector  $OD$  has components  $OA$ ,  $OB$ ,  $OC$  along the axes  $OX$ ,  $OY$ ,  $OZ$  respectively. If the direction angles which the positive direction of the vector of magnitude  $a$  makes with the positive directions of the axes  $OX$ ,  $OY$ ,  $OZ$  are  $\alpha$ ,  $\beta$ ,  $\gamma$ , the components of



the vector along the axes are,

$$X = a \cos \alpha, \quad Y = a \cos \beta, \quad Z = a \cos \gamma.$$

The components of the resultant vector are then  $\Sigma X, \Sigma Y, \Sigma Z$ , and the magnitude of the resultant vector is

$$r = [(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2]^{1/2},$$

and the direction angles of the resultant vector are

$$\cos^{-1} \frac{\Sigma X}{r},$$

$$\cos^{-1} \frac{\Sigma Y}{r},$$

$$\cos^{-1} \frac{\Sigma Z}{r}.$$

It should be noted that the direction angles of any vector are connected by the relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

This follows at once from Fig. 7, where it is seen that

$$OA^2 + OB^2 + OC^2 = OD^2,$$

and

$$OA = OD \cos \alpha, \quad OB = OD \cos \beta, \quad OC = OD \cos \gamma.$$

**8. Force a Vector Quantity.** It is an experimental fact that if a vector be used to represent a force, the magnitude of the vector representing the magnitude of the force and the direction of the vector the direction in which the force acts, then forces which act at one point may be combined by the law of vector addition. Thus, if at a point  $O$ , Fig. 8, two forces are acting which may be represented to the same scale by the vectors  $F_1$  and  $F_2$ , they are equivalent to

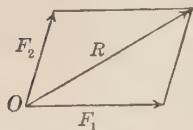


FIG. 8

a single force, represented to the same scale, by their vector sum  $R$ .

From the form of this construction this result is known as the *parallelogram of forces*.

This discussion must not be mistaken as an attempt to prove the parallelogram of forces. The law of addition of vectors has

been defined in such a way that it agrees with the experimental fact expressed by the parallelogram of forces.

For the sake of brevity a force represented by a vector  $OA$  will be referred to as the force  $OA$ .

When the resultant of a set of forces acting at a point is zero the forces are said to be a set of **balanced forces**, or to be in **equilibrium**.

Thus, in Fig. 9, where  $AC$  is equal and parallel to  $OB$ , the three forces  $OA$ ,  $OB$ , and  $CO$  are in equilibrium since their vector sum is zero.

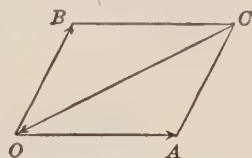


FIG. 9

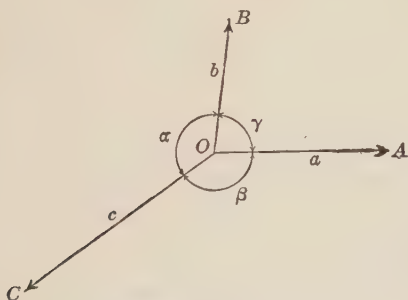


FIG. 10

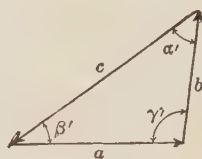


FIG. 11

**9. Lami's Theorem.** In Fig. 10 let  $OA$ ,  $OB$ ,  $OC$  be three forces in equilibrium with numerical values  $a$ ,  $b$ ,  $c$  respectively, and let the angles opposite  $a$ ,  $b$ ,  $c$ , when laid off from the same point, be  $\alpha$ ,  $\beta$ ,  $\gamma$ , respectively. Since the forces are in equilibrium, their vector sum must be zero; that is, the triangle formed of the forces must close, as in Fig. 11. In Fig. 11, we have

$$\alpha' = 180^\circ - \alpha, \quad \beta' = 180^\circ - \beta, \quad \gamma' = 180^\circ - \gamma.$$

By the law of sines,

$$\frac{a}{\sin \alpha'} = \frac{b}{\sin \beta'} = \frac{c}{\sin \gamma'},$$

or, since  $\sin (180^\circ - x) = \sin x$ ,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

Hence when three forces acting at a point are in equilibrium the force vectors form a closed triangle and the numerical values of the forces are proportional to the sines of the opposite angles when the force vectors are laid off from the same point. This statement is known as *Lami's theorem*.

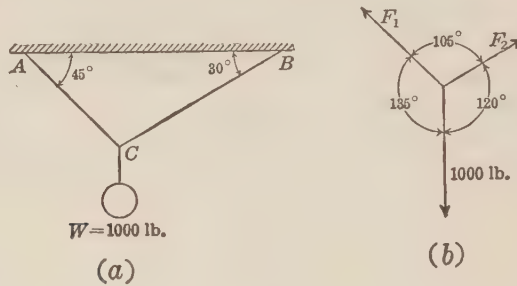


FIG. 12

**10. Illustrations of Balanced Forces.** **EXAMPLE 1.** In Fig. 12 (a) a weight of 1000 lbs. is suspended by two strings (regarded as weightless) which make angles of  $45^\circ$  and  $30^\circ$  with the horizontal. To find the forces exerted by the strings.

Here are three forces acting at the point  $C$ , the weight acting vertically downward and the two forces exerted by the strings acting in the direction of the strings. Calling the forces exerted by the strings  $CA$  and  $CB$ ,  $F_1$  and  $F_2$  respectively, we have by Lami's theorem, Fig. 12 (b),

$$\frac{F_1}{\sin 120^\circ} = \frac{F_2}{\sin 135^\circ} = \frac{1000}{\sin 105^\circ}$$

from which  $F_1 = 896$  lbs.,  $F_2 = 732$  lbs.

The forces exerted by the strings are called the *tensions* in the strings.

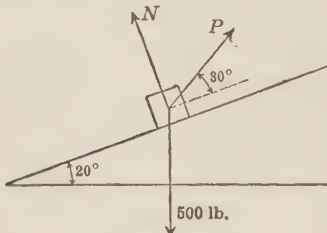


FIG. 13

**EXAMPLE 2.** In Fig. 13 is represented a weight of 500 lbs. being moved up a smooth plane inclined  $20^\circ$  to the horizontal by a force  $P$ , inclined  $30^\circ$  to the plane, which is just sufficient to move the weight. To find the value of  $P$  and the force which the plane exerts on the weight.

Since the plane is smooth, the only force that it can exert on the block is normal (perpendicular) to the plane. Call this force  $N$ .

Then by Lami's theorem,

$$\frac{P}{\sin 160^\circ} = \frac{N}{\sin 140^\circ} = \frac{500}{\sin 60^\circ},$$

from which  $P = 197$  lbs.,  $N = 371$  lbs.

# PROBLEMS

NOTE. In the solution of many problems of this book sufficiently accurate results can be obtained with the slide rule, and its use is recommended. For others, notably those on the parabola and catenary, Chapter V, logarithmic tables will be needed. The Macmillan *Logarithmic and Trigonometric Tables* will be found suitable for this purpose.

1. What two forces  $OA$  and  $OB$  in the directions shown will balance the force of 1000 lbs. in Fig. 14?

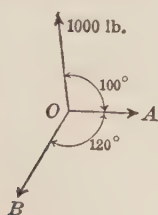


FIG. 14

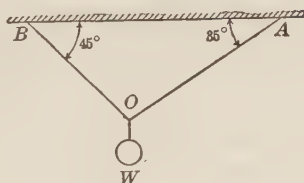


FIG. 15

2. If the tension in the cord  $OA$  in Fig. 15 is 150 lbs., what are the values of  $W$  and the tension in the cord  $OB$ ?
3. If  $W = 500$  lbs. in Fig. 15, find the tensions in  $OA$  and  $OB$ .
4. In Fig. 16, assuming that the force exerted by the inclined plane on the block is normal to the plane, find the force  $P$  acting in the direction indicated to hold the block at rest. In what direction should  $P$  act so as to be the least force that will hold the block at rest? In each case find the force that the plane exerts on the block.

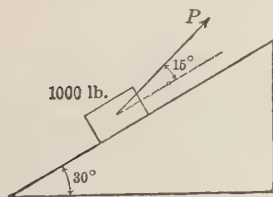


FIG. 16

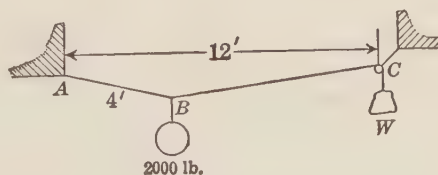


FIG. 17

5. In Fig. 17 the center of the small pulley  $C$  is on the same level as the support  $A$  of the string  $AB$ . What weight  $W$  will raise the point  $B$  to a point 1 ft. below  $AC$ ? What is then the tension in  $AB$ ?

6. A string 10 ft. long is attached at its ends to two points of support distant 9 ft. apart on the same level. It carries a weight of 25 lbs. attached to a smooth ring free to slide on the string. What is the tension in the string?

7. In Fig. 18, the weight  $W$  is supported by the cords  $AC$  and  $BC$ , of which  $AC$  is horizontal. Find the tensions in  $AC$  and  $CB$ .

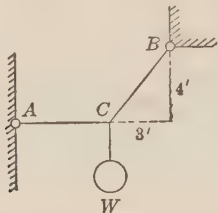


FIG. 18

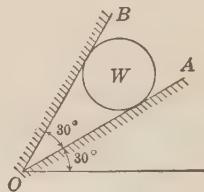


FIG. 19

8. In Fig. 19 a circular disk of weight  $W$  rests between two plane surfaces which are perfectly smooth. Assuming that the forces which the surfaces exert on the disk are normal to the planes and that these forces and the weight may be regarded as acting at the center of the disk, find the forces exerted by the surfaces against the disk.

9. In Fig. 20, what horizontal force  $P$  will hold the 10-lb. weight at a horizontal distance of 2 ft. from the point of support?

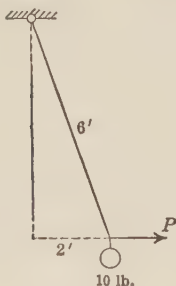


FIG. 20

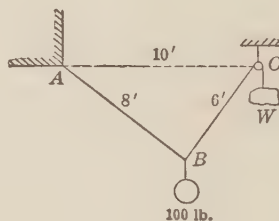


FIG. 21

10. In Fig. 21, what value of  $W$  will raise the point  $B$  to a distance of 6 ft. from  $C$  which is on the same level with  $A$ ?

11. Using the method of § 6, find the magnitude and direction of the resultant of the forces represented in Fig. 22. Check the results by a careful construction of the sum of the force vectors as described in § 3. Assuming that the result obtained by the first method is correct, what percentage of error have you made in the magnitude of the resultant by the second method? (The first method of solution is called *algebraic* and the second method *graphic*.)

12. By the algebraic method, find the magnitude and direction of a single force that will balance the forces represented in Fig. 23. Check the result by the graphic method.



13. Three ropes are attached to a weight and three men pull with the following forces in the directions indicated: 120 lbs., E.  $10^\circ$  S.; 140 lbs., E.  $15^\circ$  N.; 160 lbs., E.  $5^\circ$  N. Find the magnitude and direction of the resulting force.

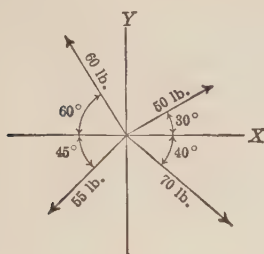


FIG. 22

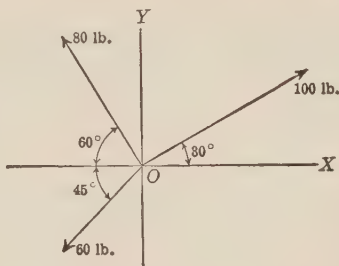


FIG. 23

14. Three ropes are attached to a weight and pass over pulleys as indicated in Fig. 24. Forces  $P_1$ ,  $P_2$ , and  $P_3$  are applied to the ropes. If  $P_1 = 100$  lbs.,  $P_2 = 120$  lbs.,  $P_3 = 80$  lbs., find the vertical force on the weight and the force tending to move the weight horizontally. Solve by the algebraic method and check by the graphic method.

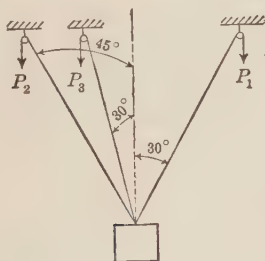
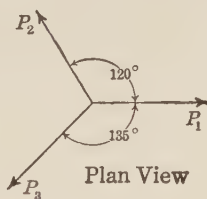


FIG. 24



15. In Fig. 24 if  $P_1 = 100$  lbs., what values should  $P_2$  and  $P_3$  have in order for the resultant force on the weight to be vertical? What weight would the three forces then just support? Would these forces raise the weight?

**11. Conditions of Equilibrium of a Set of Concurrent Forces in One Plane.** Having given a set of coplanar forces acting at one point, the resultant force  $R$  is found, as in § 6, to be

$$R = [(\Sigma X)^2 + (\Sigma Y)^2]^{1/2}.$$

If the forces are in equilibrium, the resultant force is zero; hence its rectangular components,  $\Sigma X$  and  $\Sigma Y$ , must each be zero. Conversely, if both  $\Sigma X$  and  $\Sigma Y$  vanish,  $R$  is zero and the forces are in equilibrium. Hence the algebraic conditions for

equilibrium of a set coplanar concurrent forces are

$$\Sigma X = 0 \quad \text{and} \quad \Sigma Y = 0.$$

Since only two equations are to be satisfied, only two unknown quantities can be determined from the conditions of equilibrium. These may be the components of a single force, or the magnitude and direction of a force, or the magnitudes of two forces in given directions.

Again, since, as in § 3, the resultant of the forces is obtained by vector addition, it follows that if the forces are in equilibrium the polygon obtained by adding the force vectors must close, and conversely if the polygon obtained by adding the force vectors closes, the forces are in equilibrium. Hence the graphic condition for equilibrium of a set of coplanar concurrent forces is that the force polygon close.

**12. Illustrations.** **EXAMPLE 1.** In Fig. 25, a weight of 300 lbs. is being dragged up a plane by a force  $P$  as shown. There is a friction force of 40 lbs. and a normal force  $N$  exerted by the plane on the weight. To find the values of  $P$  and  $N$  when the body has uniform motion up the plane, the forces being then in equilibrium.

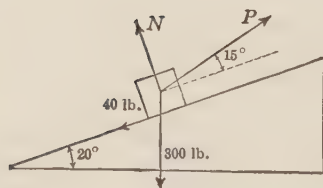


FIG. 25

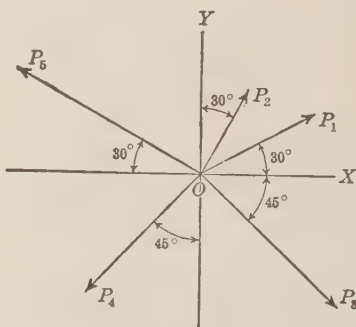


FIG. 26

Writing  $\Sigma X = 0$  and  $\Sigma Y = 0$ , taking the  $x$  axis parallel to the plane and the  $y$  axis perpendicular to the plane, we obtain the equations

$$\begin{aligned} P \cos 15^\circ - 40 - 300 \sin 20^\circ &= 0, \\ P \sin 15^\circ + N - 300 \cos 20^\circ &= 0. \end{aligned}$$

Solving these equations, we find

$$P = 147.6 \text{ lbs.}, \quad N = 243.7 \text{ lbs.}$$

EXAMPLE 2. In Fig. 26, five forces are represented. If  $P_1 = 200$  lbs.,  $P_2 = 150$  lbs.,  $P_3 = 300$  lbs., what values must  $P_4$  and  $P_5$  have if the forces are in equilibrium?

Writing the equations for equilibrium,  $\Sigma X = 0$  and  $\Sigma Y = 0$ , we have

$$\begin{aligned} 200 \cos 30^\circ + 150 \cos 60^\circ + 300 \cos 45^\circ - P_4 \cos 45^\circ - P_5 \cos 30^\circ &= 0 \\ 200 \sin 30^\circ + 150 \sin 60^\circ - 300 \sin 45^\circ - P_4 \sin 45^\circ + P_5 \sin 30^\circ &= 0. \end{aligned}$$

These reduce to

$$\begin{aligned} .707 P_4 + .866 P_5 &= 460.3, \\ .707 P_4 - .500 P_5 &= 17.8, \end{aligned}$$

from which, correct to three figures,

$$P_4 = 254 \text{ lbs.}, \quad P_5 = 324 \text{ lbs.}$$

### PROBLEMS

16. The block in Fig. 27 weighs 200 lbs. There is a friction force on the block of .20 of the normal force that the plane exerts on the block. Find the value of  $P$  in the direction shown that will just move the body up the plane, and also find the normal force between the block and plane.

17. Show that the block of Problem 16 would slide down the plane if no force were applied. Find the force  $P$  in the given direction that would just prevent the block from sliding down.

18. Replace the angle of  $20^\circ$  in Fig. 27 by a variable angle  $\theta$  and determine the normal force and the value of  $P$  to just move the body up the plane. Find the angle  $\theta$  which makes  $P$  a minimum, and determine the values of  $P$  and  $N$  for this value of  $\theta$ .

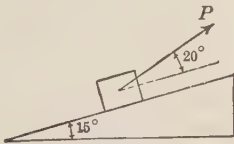


FIG. 27

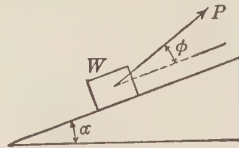


FIG. 28

19. In Fig. 28, find the value of  $P$  necessary to just move the body up the plane, given that the force of friction is  $\mu$  times the normal force between weight and plane.

20. In Fig. 28 find the force  $P$  in the direction shown that will just prevent the body from sliding down the plane.

21. From the result of Problem 19, show that the angle  $\phi$  which makes  $P$  a minimum value is  $\tan^{-1} \mu$  and show that this minimum value of  $P$  is  $W \sin (\alpha + \phi_1)$ , where  $\phi_1 = \tan^{-1} \mu$ .

22. From the result of Problem 20, show that the angle  $\phi$  which makes  $P$  a minimum is  $-\tan^{-1} \mu$  and show that this minimum value of  $P$  is  $W \sin (\alpha - \phi_1)$  where  $\phi_1 = \tan^{-1} \mu$ .

**13. Transmissibility of Forces.** It is the result of experience that the effect of a force upon the motion or equilibrium of a rigid body is not affected by changing the point of application of the force to any other point in the line of action of the force. Thus, in Fig. 29, the effect of the force is not changed by moving the point of application of the force to any other point in the line of application of the force, but the effect is changed by moving the force to any other line.

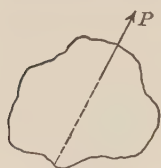


FIG. 29

**14. Body in Equilibrium under the Action of Two Forces, of Three Forces.** If a body is in equilibrium under the action of two forces, the forces must balance each other and hence must act in the same line and be equal in value and opposite in direction, that is, they must act in opposite directions and with numerically equal values in the line joining the points of application of the forces, as in Fig. 30 (a) or Fig. 30 (b). If a body is in equilibrium under the action of several forces, but these forces are applied at only two points of the body, the forces acting at each point may be combined into a single resultant and these resultants must then be equal and opposite and act in the line joining the points of application of the forces.

A body acted upon by forces at only two points is called a *two-force body*.

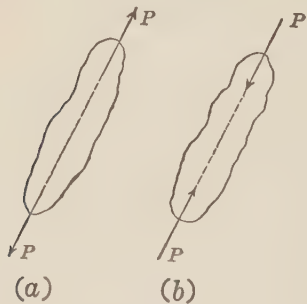


FIG. 30

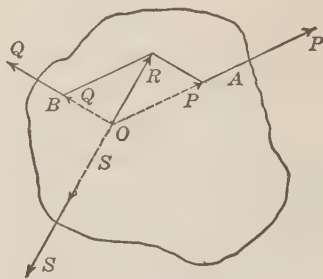
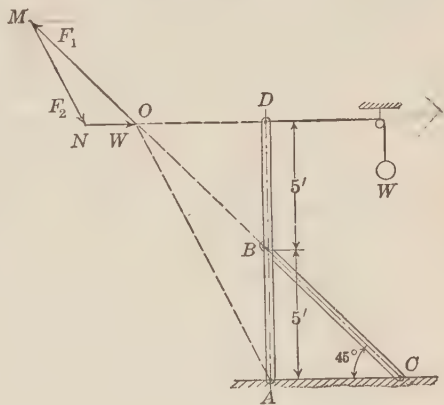


FIG. 31

If three non-parallel forces are in equilibrium, one of them must be equal and opposite to the resultant of the other two. It is therefore necessary for the lines of action of the three forces to lie in the same plane, pass through a common point, and for the vectors representing the forces to form a closed triangle. This is shown in Fig. 31. Forces  $P$  and  $Q$  acting at  $A$  and  $B$  respectively in lines intersecting at  $O$ ;  $P$  and  $Q$  may then be replaced by respectively equal forces acting at  $O$  and then be combined into their resultant  $R$ . If then a third force  $S$  is to balance  $P$  and  $Q$ , it must be equal and opposite to  $R$ . The forces  $P$ ,  $Q$ , and  $S$  therefore pass through a common point and their vectors form a closed triangle. Use of this principle can be made in solving a certain class of problems.



### 15. Illustrations. EXAMPLE

**1. In Fig. 32, a vertical post  $AD$  10 ft. high is hinged at the foot and braced by a bar  $BC$  pinned at the mid-point of  $AD$  and at the other end, and inclined at  $45^\circ$  to the vertical. From  $D$  a cord runs horizontally and, passing over a smooth pulley, carries a weight  $W$ .**

It is assumed that the weights of the members  $AD$  and  $BC$  are small compared to the weight  $W$  and can be neglected. To find the forces acting on the members  $AD$  and  $BC$ .

Since  $BC$  is acted on by forces at only two points, its ends  $B$  and  $C$ , it is a two-force body and the forces acting on it act in the line  $BC$ . Accepting the principle that when two bodies are in contact the forces that they exert on each other are equal and opposite, we may then say that the member  $BC$  exerts on  $AD$  a force along the line  $BC$ . The cord at  $D$  exerts on  $AD$  a horizontal force equal to the weight  $W$ . Also at  $A$  the pin exerts a force on  $AD$ . Therefore  $AD$  is in equilibrium under the action of three forces—a horizontal force  $W$  acting at  $D$ , a force along  $BC$  acting at  $B$ , and a force acting at the pin  $A$ . By the preceding article, the lines of action of these forces must pass through a common point, and the force vectors must form a closed triangle.

FIG. 32



Hence if the horizontal line through  $D$  is extended to meet the line through  $B$  and  $C$ , the point of intersection,  $O$ , of two of the forces is obtained and the third force, the pin-force at  $A$ , must pass through this point. Hence the pin-force acting on  $AD$  at  $A$  is along the line  $AO$ . If then a triangle  $OMN$  is drawn with sides parallel to  $OD$ ,  $BC$ , and  $AO$ , with the side parallel to  $OD$  equal to  $W$ , the other two sides,  $OM$  and  $MN$ , represent the values of the force in  $BC$  and the pin force at  $A$ . The values of the last two forces may then be scaled in terms of the weight  $W$ . Since the direction of the force  $W$  is known, the directions of the forces  $OM$  and  $MN$  must be as indicated in the figure, so that the three forces have a resultant zero.

EXAMPLE 2. In Fig. 33 is shown a sketch of a frame supporting a weight  $W$ . It is pinned at  $A$  and held in position by a horizontal member  $GF$ , pinned at  $G$  and  $F$ . The weights of the members of the

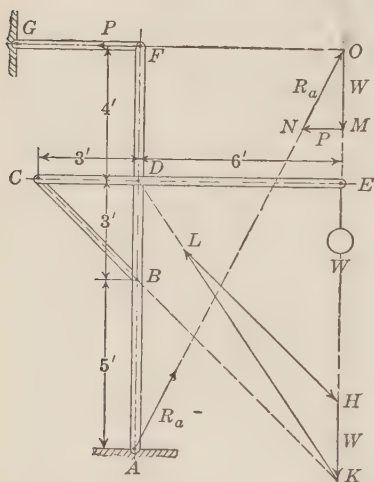


FIG. 33

frame may be neglected in comparison with the weight  $W$ . The member  $GF$ , being acted upon at only two points,  $G$  and  $F$ , is a two-force member, and it therefore exerts a force along  $FG$  upon the member  $AF$ . The frame is therefore acted upon by the weight  $W$ , a horizontal force in the line  $FG$ , and a force exerted by the pin at  $A$  in an unknown direction. The frame is therefore a three-force body in equilibrium; hence the lines of action of the three forces must pass through a common point, and the vectors representing the forces must form a closed triangle. The lines of action of  $W$  and the force in  $FG$  are

therefore extended to meet in  $O$ . Since the line of action of the third force, acting at  $A$ , must pass through this point, the direction of this force is along  $AO$ . The values of the two unknown forces are then found by forming a closed triangle of  $W$  and two forces parallel to  $FG$  and  $AO$  respectively. This may be done by laying off vertically from  $O$  a vector to represent  $W$  and then drawing the other two sides,  $MN$  and  $NO$ , of the triangle parallel to  $FG$  and  $AO$  respectively. The vectors  $MN$  and  $NO$  then represent the force in  $FG$  and the pin-force, or reaction, at  $A$ . The directions of these forces must be as shown in order that the force triangle close.

The figure may be drawn carefully and the vectors scaled to obtain



the values of the forces, or the force triangle may be solved by comparison with a similar triangle whose sides are known. Such a triangle is  $AFO$ , and the proportion may be written

$$\frac{MN}{FO} = \frac{NO}{AO} = \frac{W}{AF}$$

from which we find

$$MN = 6W/12 = .5W, \quad NO = 6\sqrt{5}W/12 = 1.118W.$$

Next, considering the member  $CE$ , it is seen to be a three-force body in equilibrium under the action of  $W$ , a force in the two-force member  $CB$  and a force exerted by the pin at  $D$ . The force triangle  $HKL$  for this member is shown in the figure, in which  $LH = 2.83W$  is the force in  $BC$ , and  $KL = 3.61W$  is the pin-force at  $D$ , that is, the force exerted on the member  $CE$  by the pin at  $D$ .

A shorter method of solving problems of this kind will be available after the discussion of moments of forces.

### PROBLEMS

Solve the following problems by graphical construction and scaling the forces.

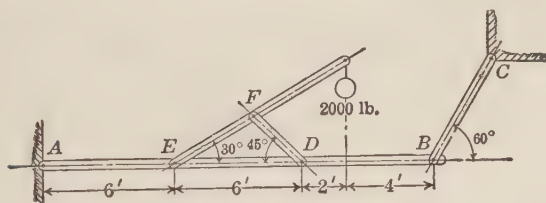


FIG. 34

23. In Fig. 34, a pinned structure, find the tension in  $BC$  and the pin-force at  $A$ ; also the compression in  $DF$  and the pin-force at  $E$ .

24. In Fig. 35, a pinned structure, the beam  $AB$  rests on a smooth roller at  $B$ , so that the reaction of the support is vertical at that point; find the reactions at  $B$  and  $A$ . Find also the tension in  $DF$  and the pin-force at  $C$ .

(SUGGESTION. First combine the two forces in the rope into a single force at  $E$ .)

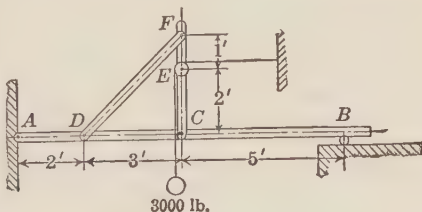


FIG. 35

25. The rectangular block in Fig. 36 weighs 800 lbs. Assuming that it is prevented from slipping at  $A$ , find the force  $P$  necessary to tip it up when (a)

$\theta = 15^\circ$ , (b)  $\tan \theta = 2/3$ , (c)  $\theta = 45^\circ$ . Assume the weight of the block to act along the center line.

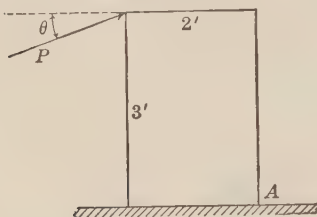


FIG. 36

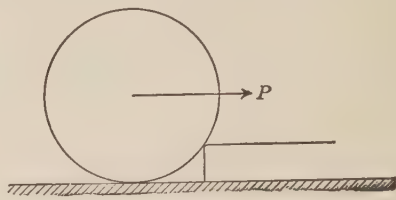


FIG. 37

26. The wheel in Fig. 37 is 3 ft. in diameter and weighs 200 lbs. What horizontal force  $P$ , acting at its center, will start it over the obstruction 8 inches high? In what direction should the force  $P$  at the center be applied in order for it to be a minimum? What then is its value?

**16. Moment of a Force.** The product of a force and the length of the perpendicular from any point to the line of action of the force, taken with the proper sign, is called the *moment of the force* with respect to that point. If the force be thought of as applied to a body pivoted at the given point, it would tend to turn the body about that point. If this tendency to rotation is counter-clockwise, the moment is called positive and the plus sign is used; if clockwise, the moment is negative and the minus sign is used. Thus, in Fig. 38, the moment of  $P$  with respect to  $O$  is  $Pd$ , and the moment of  $P$  with respect to  $O'$  is  $-Pd'$ .

Since  $\frac{1}{2}Pd$  is equal to the area of the triangle  $OAB$ , the value of the moment of a force with respect to a point is numerically equal to twice the area of the triangle which has the force vector as base and the given point as vertex, the sign being taken as positive or negative according as the tendency to rotation is counter-clockwise or clockwise.

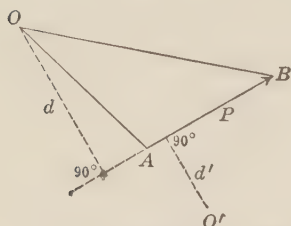


FIG. 38

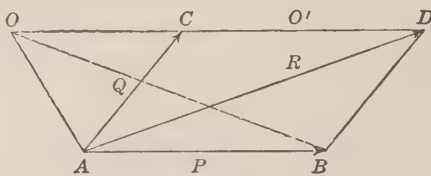


FIG. 39

**17. Varignon's Theorem of Moments.** *The sum of the moments of two intersecting forces with respect to any point in their plane is equal to the moment of their resultant with respect to that point.*

PROOF. Let two intersecting forces,  $P$  and  $Q$ , meeting at the point  $A$ , have a resultant  $R$ , Fig. 39, and let  $O$  be any point in the plane of the forces. Through  $O$  draw a line parallel to one of the forces, as  $P$ , cutting the line of action of  $Q$  in  $C$ , and let the scale of forces be chosen so that  $AC$  represents  $Q$ . Lay off  $P = AB$  to the same scale and their resultant  $R$  is then represented by the diagonal  $AD$ .

By § 16, we may write

$$\text{mom } P + \text{mom } Q = 2 (\text{area } ABO + \text{area } ACO).$$

But area  $ABO = \text{area } ACD$ , since the triangles have equal bases and equal altitudes. Therefore

$$2(\text{area } ABO + \text{area } ACO) = 2 \text{ area } ADO = \text{mom } R,$$

and hence

$$\text{mom } P + \text{mom } Q = \text{mom } R.$$

Any other point, as  $O'$ , may be chosen and a similar proof be given if proper regard be given to signs.

A second proof, extended to include more than two forces, may be given as follows.

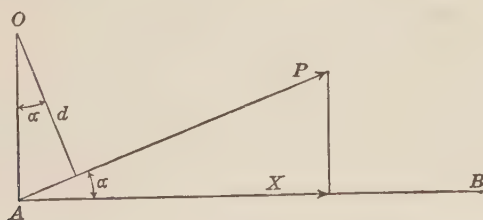


FIG. 40

First consider a single force  $P$  acting at a point  $A$  (Fig. 40). Let  $O$  be any point distant  $d$  from the line of action of  $P$ . Draw  $AB$  perpendicular to  $AO$  and let  $X$  be the projection of  $P$  upon  $AB$ . Then by similar triangles,

$$P/AO = X/d, \quad \text{or} \quad P \cdot d = AO \cdot X;$$

that is, the moment of  $P$  with respect to  $O$  is equal to  $AO$  times the projection of  $P$  upon a line perpendicular to  $AO$ .

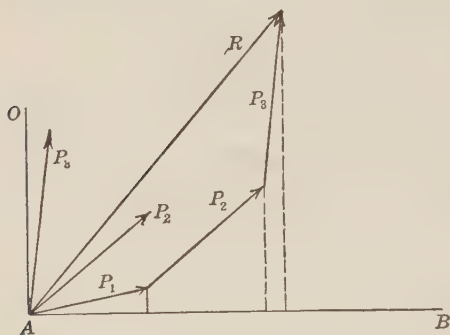


FIG. 41

Consider now several forces in the same plane all acting at  $A$  (Fig. 41). Let the forces be  $P_1, P_2, P_3$ , etc., and let their resultant be  $R$ . From the figure it is clear that the sum of the projections of  $P_1, P_2, P_3$ , etc. upon  $AB$  is equal to the projection of  $R$  upon  $AB$ . Making use of the equation

$P \cdot d = AO \cdot X$  we may then write

$$\begin{aligned} \Sigma P \cdot d &= \Sigma AO \cdot X = AO \Sigma X = AO (\text{projection of } R \text{ on } AB) \\ &= \text{mom } R. \end{aligned}$$

In other words, *the sum of the moments of any number of co-planar, concurrent forces with respect to any point in their plane is equal to the moment of their resultant with respect to that point.*

In this proof, the forces and the point  $O$  were chosen so that the moment of each force with respect to  $O$  is positive. However, the proof is valid at each step if positive and negative directions be given to distances along  $AB$  such that a positive direction would represent counter-clockwise rotation about  $O$ , and negative direction clockwise rotation, and if the projection of a force upon  $AB$  means the distance on  $AB$  from the projection of the initial point of the force to the projection of the terminal point, with proper sign.

**18. Resultant of Two Parallel Forces.** (a) When the forces act in the same direction. Let the forces be  $P_1$  and  $P_2$ , the points of application being  $A$  and  $B$ , Fig. 42 (a). Resolve  $P_1$  into components  $F$  and  $Q_1$ , where  $F$  is any force along  $AB$ . Then resolve  $P_2$  into components  $F'$  and  $Q_2$ , where  $F'$  is equal and opposite to  $F$  and in the same line. The forces  $F$  and  $F'$  then

annul each other, and there remain only  $Q_1$  and  $Q_2$ , whose resultant may be found by moving them in their lines of action to

their point of intersection, and taking their vector sum. From the manner in which  $Q_1$  and  $Q_2$  were obtained it is evident (see Fig. 42 (b)) that their resultant is equal to  $P_1 + P_2$ . Moreover, if this resultant cuts  $AB$  in  $M$ , we have, from similar triangles,

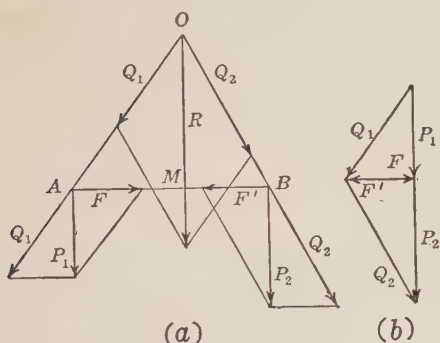


FIG. 42

$$\frac{AM}{MO} = \frac{F}{P_1}, \quad \text{and} \quad \frac{MB}{MO} = \frac{F}{P_2},$$

from which we get

$$\frac{AM}{MB} = \frac{P_2}{P_1}.$$

Hence, the resultant of two parallel forces acting in the same direction is equal to their sum, acts parallel to them and divides any line joining two points on the lines of the forces in the inverse ratio of the forces.

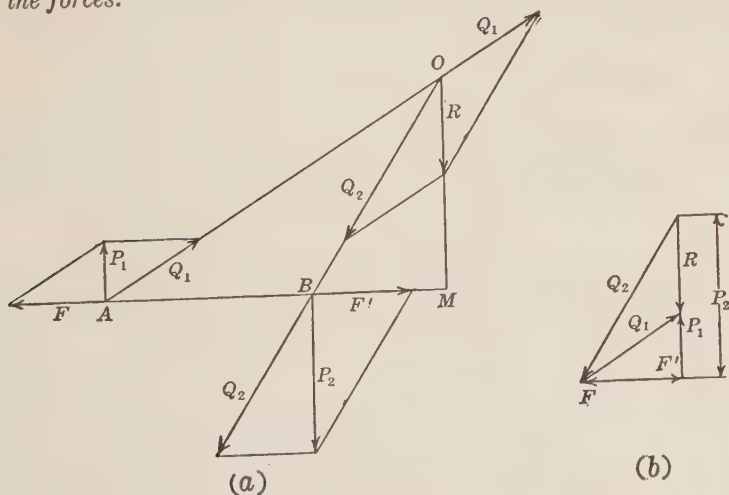


FIG. 43

(b) When the forces act in opposite directions and are numerically unequal. The construction is shown in Fig. 43. The unequal forces,  $P_2 > P_1$ , are resolved into components  $F$  and  $Q_1$ , and  $F'$  and  $Q_2$ , where  $F$  and  $F'$  annul each other. The resultant of  $Q_1$  and  $Q_2$  is then seen to be  $P_2 - P_1$  (see Fig. 43 (b)), and to divide the line  $AB$  externally in such a way that

$$\frac{AM}{BM} = \frac{P_2}{P_1}.$$

Hence the resultant of two parallel, numerically unequal, forces acting in opposite directions is a force equal to their numerical difference, acts in the direction of the larger force and divides any line joining two points on the lines of the forces externally in the inverse ratio of the forces.

**19. Moment of the Resultant of Two Parallel Forces.** Referring to either Fig. 42 (a) or Fig. 43 (a), we have, by § 17, taking moments with respect to any point in the plane,

$$\begin{aligned}\text{mom } P_1 &= \text{mom } Q_1 + \text{mom } F, \\ \text{mom } P_2 &= \text{mom } Q_2 + \text{mom } F' .\end{aligned}$$

Since  $F$  and  $F'$  are equal and opposite and act in the same line, the sum of their moments with respect to any point is zero. Therefore

$$\text{mom } P_1 + \text{mom } P_2 = \text{mom } Q_1 + \text{mom } Q_2 = \text{mom } R.$$

Hence the sum of the moments of two parallel forces with respect to any point in their plane is equal to the moment of their resultant with respect to that point.

**20. Couple. Moment of a Couple.** Two parallel, numerically equal forces, acting in opposite directions, are called a *couple*. Thus, in Fig. 44, where  $P$  and  $P'$  are parallel and numerically equal, the forces form a couple.

If any point  $O$  in the plane of the couple is chosen, the sum of the moments of the forces of the couple with respect to the point is equal to the product of one of the forces and the perpendicular



distance between the forces, taken with the proper sign. For, in Fig. 44, taking moments with respect to  $O$ , we have

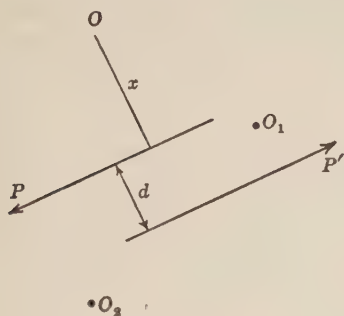


FIG. 44

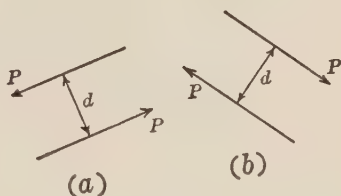


FIG. 45

$$\text{mom } P' + \text{mom } P = P'(d + x) - Px = P'd + P'x - Px = Pd.$$

If any other point, as  $O_1$  or  $O_2$ , be taken, the same result is found to hold. If the directions of  $P$  and  $P'$  are reversed, the only change will be to change the sign of  $Pd$ .

The product of one of the forces of a couple and the perpendicular distance between the forces, taken with the proper sign, is called the **moment** of the couple. The plus sign is used when the arrows of the force vectors indicate counter-clockwise rotation, and the minus sign when they indicate clockwise rotation. Thus, in Fig. 45 (a), the moment of the couple is  $Pd$ , and in Fig. 45 (b) the moment of the couple is  $-Pd$ .

**21. Combination of Coplanar Couples.** Let  $(P, P')$  and  $(Q, Q')$ , Fig. 46, be the forces forming two couples of moments  $M_1$  and  $M_2$  respectively.

(a) If  $P$  and  $Q$  are not parallel, extend their lines of action until they intersect, move  $P$  and  $Q$  to this point, and combine them into a force

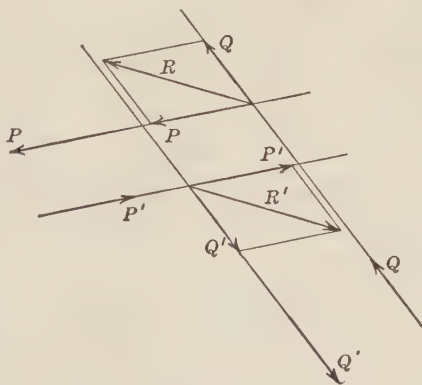


FIG. 46

$R$ . Similarly combine  $P'$  and  $Q'$  into  $R'$ . Then  $R$  and  $R'$  are equal numerically, parallel, and opposite in direction, as is easily shown. They therefore form a couple. Let the moment of this couple be  $M$ . By § 20, taking moments with respect to any point in the plane, we have

$$\begin{aligned} M &= \text{mom } R + \text{mom } R' \\ &= \text{mom } P + \text{mom } Q + \text{mom } P' + \text{mom } Q' \\ &= (\text{mom } P + \text{mom } P') + (\text{mom } Q + \text{mom } Q') \\ &= M_1 + M_2. \end{aligned}$$

(b) If the lines of action of the forces forming the couples are parallel, as in Fig. 47, one force from each couple may be combined into a single force  $R = P + Q$ , and the other two forces combined into the single force  $R' = P' + Q'$  (§ 18). Then  $R$  and  $R'$  form a couple with moment

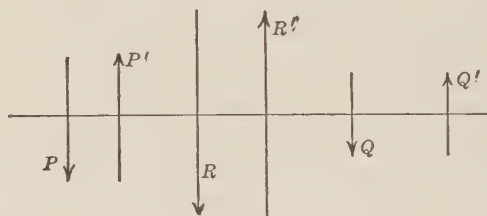


FIG. 47

$$M = \text{mom } R + \text{mom } R' \quad (\S 20)$$

But

$$\text{mom } R = \text{mom } P + \text{mom } Q$$

and

$$\text{mom } R' = \text{mom } P' + \text{mom } Q' \quad (\S 19)$$

Hence

$$M = (\text{mom } P + \text{mom } P') + (\text{mom } Q + \text{mom } Q') = M_1 + M_2.$$

Therefore, in any case, *two couples in the same plane are equivalent to a single couple whose moment is the sum of the moments of the two couples.*

By an evident extension of the method, it follows that *any number of couples in the same plane are equivalent to a single couple in that plane with a moment equal to the sum of the moments of the given couples.*

**22. Equivalent Couples.** In Fig. 48  $L$  and  $L'$  are lines of action of forces  $P$  and  $P'$  which form a couple of moment  $M$ , and  $N$  and  $N'$  are two parallel lines intersecting  $L$  and  $L'$  in  $A$  and  $B$ .

The points  $A$  and  $B$  may be taken as points of application of the forces  $P$  and  $P'$ . In the line  $AB$ , two equal and opposite forces  $F$  and  $F'$  may be put in, since they amount to a zero force, and their values may be so determined that when combined with  $P$  and  $P'$ , respectively, the resultant forces,  $Q$  and  $Q'$ , will fall in the lines  $N$  and  $N'$ . We thus replace the couple formed of  $P$  and  $P'$  by one formed of  $Q$  and  $Q'$ .

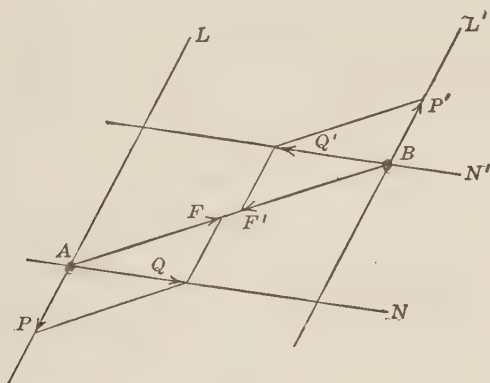


FIG. 48

Let  $M'$  be the moment of the new couple. Then, by § 20,

$$M' = \text{mom } Q + \text{mom } Q',$$

the moments being taken with respect to any point in the plane. But, by § 17,

$$\begin{aligned} \text{mom } Q &= \text{mom } P + \text{mom } F, \\ \text{mom } Q' &= \text{mom } P' + \text{mom } F'. \end{aligned}$$

Since  $F$  and  $F'$  are two equal and opposite forces in the same line,

$$\text{mom } F + \text{mom } F' = 0.$$

Hence

$$\text{mom } Q + \text{mom } Q' = \text{mom } P + \text{mom } P',$$

or,

$$M' = M.$$

It follows that the original couple has been replaced by another

couple in the same plane in any parallel lines intersecting the lines of the first couple, the new couple having the same moment as the first. By a second application of the method, the couple  $(Q, Q')$  could be replaced by a couple in lines parallel to  $P$  and  $P'$ . We may therefore say that *any couple may be replaced by another couple in the same plane, provided only that their moments are equal.*

Two couples in the same plane with equal moments are therefore called **equivalent couples**.

### 23. Combination of a Couple and a Force in the Same Plane.

In Fig. 49,  $(P, P')$  form a couple of moment  $M$ , and  $Q$  is a force in the same plane. By § 22, the couple may be replaced by any other couple in the plane having an equal moment. The couple

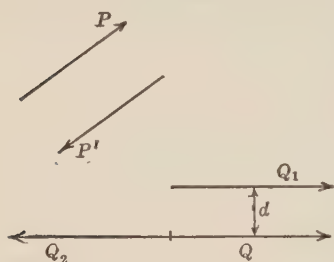


FIG. 49

$(P, P')$  may then be replaced by the couple  $(Q_1, Q_2)$  so placed that  $Q_2$  is equal and opposite to  $Q$ , and  $Q_1$  is at a distance  $d = M/Q$  from  $Q$ . The forces  $Q$  and  $Q_2$  then annul each other, and there is left only the force  $Q_1$ . Therefore a couple and a force in the same plane are equivalent to a single force, equal in magnitude and

direction to the original force, but placed at a distance from it equal to the moment of the given couple divided by the value of the force. The sign of the moment of the couple determines on which side of the given force the resultant force appears.

Since the combination of a couple and a force is always a force, it follows that a force and a couple cannot balance each other.

**24. A Force is Equivalent to a Force and a Couple.** In Fig. 50  $P$  is any force. At any point  $O$  two equal and opposite forces numerically equal to and parallel to  $P$  may be applied, and the three forces will be equivalent to the original force  $P$ . They may, however, be regarded as a single force,  $P_1$ , equal in magnitude and direction to the original force  $P$ , but acting at  $O$ , and a couple

( $P, P_2$ ), with moment equal to  $Pd$ , or the moment of the original force with respect to  $O$ .

Hence a force is equivalent to an equal force acting at any point  $O$ , and a couple in the plane of the given force and the point, with a moment equal to the moment of the original force with respect to the point.

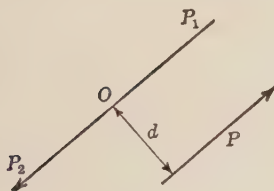


FIG. 50

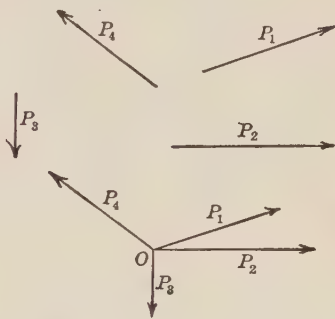


FIG. 51

**25. Resultant of any Number of Coplanar Forces.** Let  $P_1, P_2, P_3$ , etc. (Fig. 51) be any set of forces in one plane, and  $O$  be any point in the plane. By § 24, each force may be replaced by a numerically equal and parallel force acting at  $O$  and a couple whose moment is equal to the moment of the force with respect to  $O$ . The whole set of forces may therefore be replaced by a set of forces equal in magnitude and direction to the given forces, but all acting at  $O$ , and a set of couples in the plane whose moments are respectively equal to the moments of the given forces with respect to  $O$ .

The forces at  $O$  may then be resolved into  $x$  and  $y$  components, and then combined into a single force  $R = [(\Sigma X)^2 + (\Sigma Y)^2]^{1/2}$ , by § 6; and the couples may be combined into a single couple whose moment is equal to the sum of the moments of the given forces with respect to  $O$  (§ 21).

Hence any set of forces in one plane are equivalent to a single force  $R = [(\Sigma X)^2 + (\Sigma Y)^2]^{1/2}$ , acting in the plane at any chosen point, and a couple whose moment is  $M = \Sigma Pd$ , where  $\Sigma X$  and  $\Sigma Y$  are the sum of the components of the forces parallel respectively to two rectangular axes, and  $\Sigma Pd$  is the sum of the moments of the given forces with respect to the chosen point.

**26. Further Reduction of a Set of Coplanar Forces.** Having reduced the set of coplanar forces of § 25 to a single force  $R$  acting at  $O$  and a single couple of moment equal to  $M = \Sigma Pd$ , these may be replaced by the method of § 23 by a single force equal in magnitude and direction to  $R$  acting at a distance  $M/R$  from  $O$ . Hence, a set of coplanar forces is equivalent to a single force with components equal to  $\Sigma X$  and  $\Sigma Y$  of the given forces, acting in a line of the plane such that its moment with respect to any point of the plane is equal to the sum of the moments of the given forces with respect to that point.

If  $\Sigma X$  and  $\Sigma Y$  are both zero, the proof does not hold, the given set of forces in this case being equivalent to a couple, unless also  $\Sigma Pd = 0$ , in which case the set of forces are in equilibrium.

**27. Conditions of Equilibrium of a Set of Coplanar Forces.** If the forces of § 25 are in equilibrium, since the resultant force and the resultant couple cannot balance each other, § 23, the moment of the resultant couple must be zero, and the components of the resultant force must be zero; that is,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Pd = 0.$$

Conversely, if these equations hold, the resultant force and the resultant couple are both zero and the forces are therefore in equilibrium.

The point about which moments are taken may be any point in the plane and the directions of  $x$  and  $y$  may be any directions. Any number of equations of the form of those above may be written, but since the three equations written above are sufficient conditions for equilibrium, it follows that any others that are written are not independent of these three. Hence only three independent equations for the equilibrium of a set of coplanar forces can be written. It is not necessary that the three independent equations be in the form written above. They may be those obtained by equating to zero the sum of the moments of the forces with respect to three points of the plane not in the same straight line.

When a rigid body acted upon by a set of coplanar forces is at rest, the set of forces acting on it must be in equilibrium.



**28. Illustrations.** EXAMPLE 1. In Fig. 52, the four forces shown are to be replaced by an equivalent set of forces consisting of a force acting at  $O$  and a couple composed of forces acting in the lines  $L$  and  $L'$

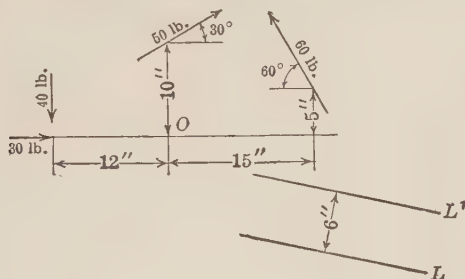


FIG. 52

$L'$ , 6 in. apart. We have

$$\begin{aligned}\Sigma X &= 30 + 50 \cos 30^\circ - 60 \cos 60^\circ \\ &= 43.30 \text{ lbs.}\end{aligned}$$

$$\begin{aligned}\Sigma Y &= -40 + 50 \sin 30^\circ + 60 \sin 60^\circ \\ &= 36.96 \text{ lbs.}\end{aligned}$$

$$\begin{aligned}\Sigma Pd &= 40 \cdot 12 + 60 \sin 60^\circ \cdot 15 + 60 \cos 60^\circ \cdot 5 - 50 \cos 30^\circ \cdot 10 \\ &= 976.4 \text{ lbs. in.}\end{aligned}$$

Hence

$$R = [(43.30)^2 + (36.96)^2]^{1/2} = 56.93 \text{ lbs.,}$$

the resulting force acting at  $O$ . The direction of  $R$  is given by

$$\tan^{-1} \frac{\Sigma Y}{\Sigma X} = 40^\circ 29'$$

with the  $x$  axis. The values of the forces in the lines  $L$  and  $L'$  must be  $976.4/6$ , or 162.7 lbs., and must be directed to give a positive moment.

If the single resultant force of the three given forces is desired, it is seen by § 26 to be the above value of  $R$  in magnitude and direction but acting in a line at a distance

$$\frac{\Sigma Pd}{R} = \frac{976.4}{56.93} = 17.15 \text{ in.}$$

from  $O$  and so placed as to give a positive moment as in Fig. 53.

EXAMPLE 2. In Fig. 54, a weight of 5000 lbs. is supported on a frame pinned at  $A$  and held by a two-force member  $BC$ . It is desired to find the values of the tension in  $BC$  and the pin-force at  $A$ , and also the compression in  $DF$  and the pin-force at  $E$ .

Regarding the whole frame as a rigid body, the forces acting on it, namely, the weight of 5000 lbs., the pin-force, or *reaction*, at *A*, and the force acting at *B* along *BC* (the tension in *BC*) are in equilibrium. Representing the horizontal and vertical components of the force that

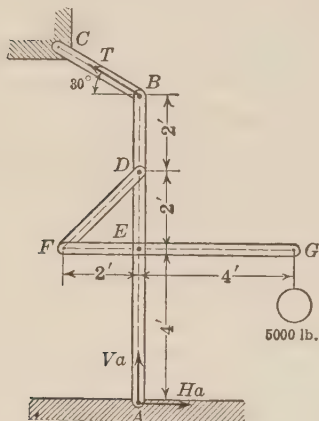


FIG. 54

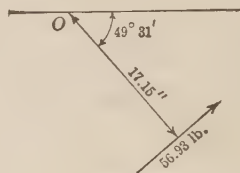


FIG. 53

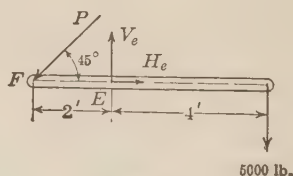


FIG. 55

the pin at *A* exerts on the frame by  $H_a$  and  $V_a$ , and the force exerted on the frame by the tie-bar *BC* by  $T$ , we may write the following equations of equilibrium, moments being taken with respect to (or about) *A*,

$$\Sigma X = H_a - T \cos 30^\circ = 0,$$

$$\Sigma Y = V_a + T \sin 30^\circ - 5000 = 0,$$

$$\Sigma Pd = T \cos 30^\circ \cdot 8 - 5000 \cdot 4 = 0.$$

Solving these equations we get

$$T = 2887 \text{ lbs.}, \quad H_a = 2500 \text{ lbs.}, \quad V_a = 3557 \text{ lbs.}$$

The reaction at *A* is therefore

$$R_a = (H_a^2 + V_a^2)^{1/2} = 4348 \text{ lbs.},$$

acting at an angle

$$\tan^{-1} (3557/2500) = 54^\circ 54'$$

with the horizontal.

Again, if the forces acting on the member *FG* be considered, they may be represented as shown in Fig. 55. The three equations of equilibrium may be written in the form

$$H_e - P \cos 45^\circ = 0,$$

$$V_e - P \sin 45^\circ - 5000 = 0,$$

$$P \cos 45^\circ \cdot 2 - 5000 \cdot 4 = 0,$$

from which we find the values,

$$P = 14,142 \text{ lbs.}, \quad H_e = 10,000 \text{ lbs.}, \quad V_e = 15,000 \text{ lbs.}$$

Hence the compression in  $FD$  is 14,142 lbs. and the pin-force at  $E$  is 18,028 lbs., acting at  $\tan^{-1} 1.5 = 56^\circ 19'$  with  $EG$ .

PROBLEMS

27. In Fig. 56, the coordinates of the points of application of three of the forces shown are  $A_1 (10, 5)$ ,  $A_2 (8, -6)$ ,  $A_3 (-6, -4)$ , and the 50-lb. force acts upward. Find the force at the origin and a couple acting in the lines  $L$  and  $L'$  that are equivalent to the four forces. Find also a single force in magnitude, direction, and line of action, which is equivalent to the four forces. The coordinates of the points are in inches.

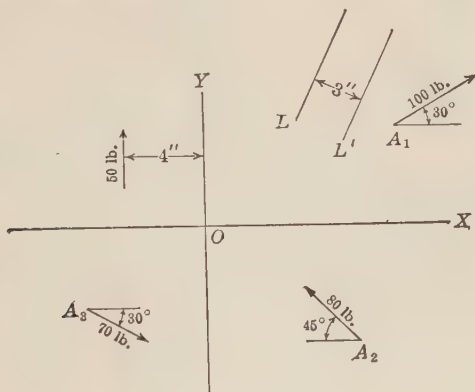


FIG. 56

28. Find the tension in  $ED$  and the pin-force at  $A$  acting on the frame of Fig. 57. Find also the compression in  $BG$  and the pin-force at  $C$ .

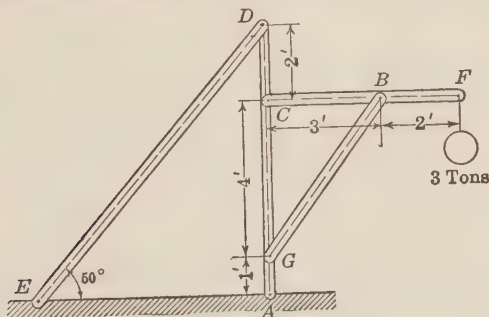


FIG. 57

29. Neglecting the weights of the members of the frame in Fig. 58, find the tension in  $BC$  and the horizontal and vertical components of the force that the pin at  $A$  exerts on the frame. Find also the forces in the members  $CE$  and  $DE$ .

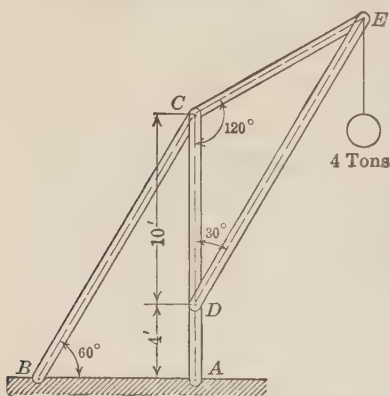


FIG. 58

30. Prove that if the sum of the moments of a set of coplanar forces about each of two points in the plane is zero, the sum of the components of the forces perpendicular to the line joining the two points is zero.

31. Prove that if the sum of the moments of a set of coplanar forces about each of three points of the plane not in the same straight line is zero, the forces are in equilibrium.

32. The block of Fig. 59 is prevented from sliding by a small peg at  $A$ . What force  $P$  as shown will just tip it up on the corner at  $A$ ? What is the reaction at  $A$  just as it starts? Assume the weight to act at the center of the block.  $W = 1600$  lbs.

33. In Fig. 60, the block, weighing 200 lbs., is on the point of tipping about the edge  $AB$  under the action of its weight, an 800-lb. force acting horizontally, and the force  $P$  in a vertical plane in a direction  $20^\circ$  below the horizontal. Find the value of  $P$  and the components of the reaction at the edge  $AB$ , assuming all of the forces to lie in one plane.

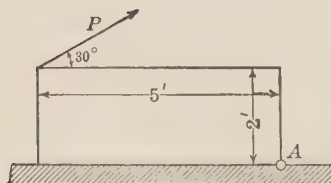


FIG. 59

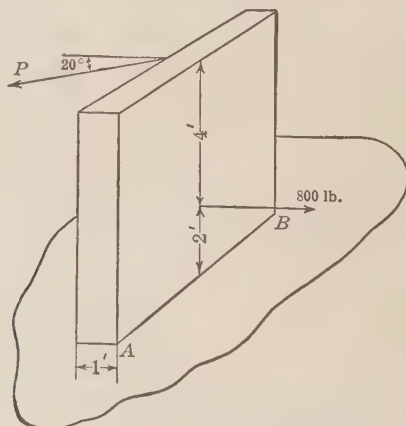


FIG. 60

34. In Fig. 61, show that the tensions in the rope passing around the pulley at  $B$  are equivalent to two forces, each of 200 lbs., applied at the center of the pulley and having the directions of the rope tangent to the pulley at  $B$ . Then find the components of the reaction at  $A$  and the tension in  $CD$ .

35. The structure shown in Fig. 62 is composed of four members pinned at  $A$ ,  $B$ ,  $C$ , and  $D$ ; and carries loads as shown. Neglecting the weights of the members, find the tension in the horizontal member  $CD$  and the components of the reaction at  $A$ . Find also the forces exerted by each of the members  $AB$ ,  $BC$ ,  $CA$ .

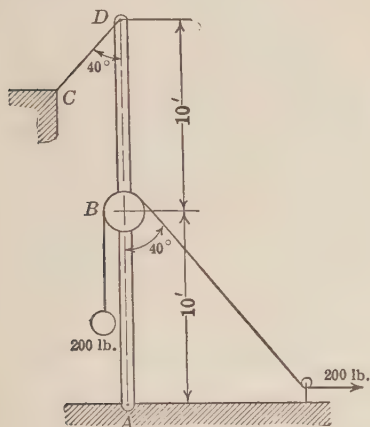


FIG. 61

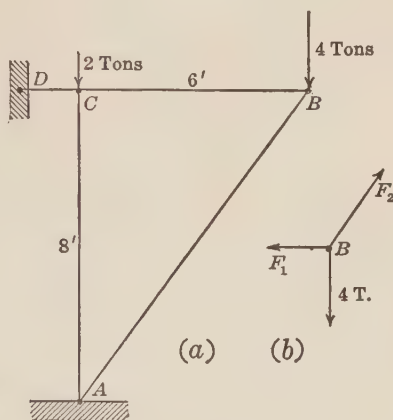


FIG. 62

(SUGGESTION. At the point  $B$  the forces are as shown in Fig. 62 (b), where  $F_1$  and  $F_2$  represent the forces in  $BC$  and  $BA$ , respectively.)

(36) In Fig. 63, the eight members are fastened together by pins at the joints. The member  $BC$  is horizontal. Neglecting the weights of the members, find the tension in  $BC$  and the horizontal and vertical components of the reaction at  $A$ . Find also the forces in each of the seven other members of the truss and state whether they are in tension or in compression.

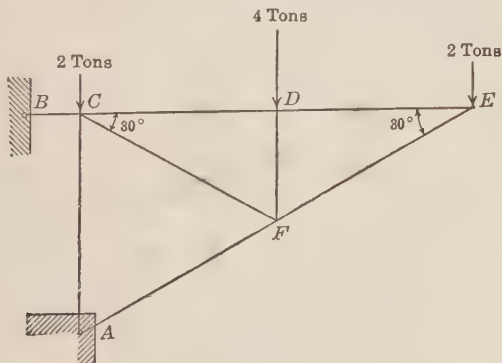


FIG. 63

(SUGGESTION. In finding the forces in the members, consider first the forces acting at  $E$ , next those at  $D$ , etc. Fig. 64 shows how the forces act at some of the joints).

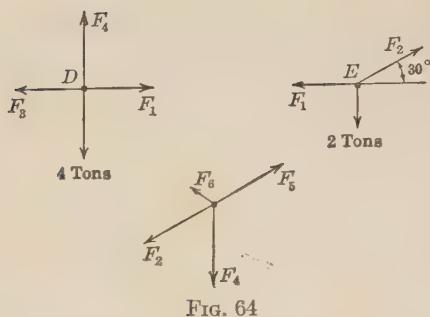


FIG. 64

37. The truss of Fig. 65 is composed of seven pin-connected members, and carries the loads shown. The truss is hinged at the left support and rests upon a smooth roller at the right. Find the reaction at the right support and the horizontal and vertical components of the reaction at the left support. Then find the forces in all members of the truss, and state whether they are in tension or in compression.

38. In Fig. 66, when the force  $P$  is applied, there is a tangential friction force at  $B$  equal to  $1/4$  of the normal force there. By considering the forces acting on the drum, find what tangential force at  $B$  is necessary to just prevent

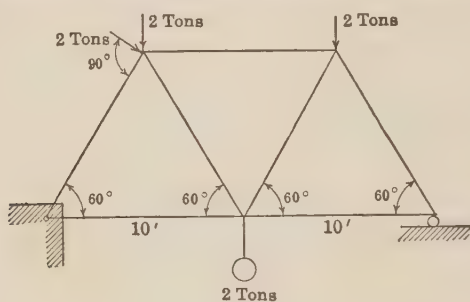


FIG. 65

the motion of the 500-lb. weight and find also the reaction at the axle. Assuming the depth of the bar to be 8 in., and the pivot at  $A$  to be at the center of the depth, find the force  $P$  and the reaction at  $A$ .

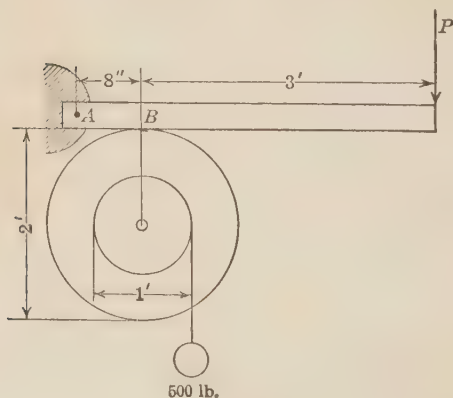


FIG. 66

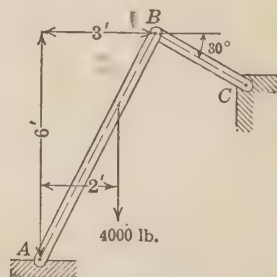


FIG. 67



39. In Fig. 67 the member  $BC$  makes  $30^\circ$  with the horizontal; find the compression in  $BC$  and the components of the reaction at  $A$ .

40. In Fig. 68 (a) the weights are hung from the middle points of the members  $AB$  and  $BC$ ; find the reactions at  $A$ ,  $B$ , and  $C$ . Angle at  $A = 60^\circ$ .

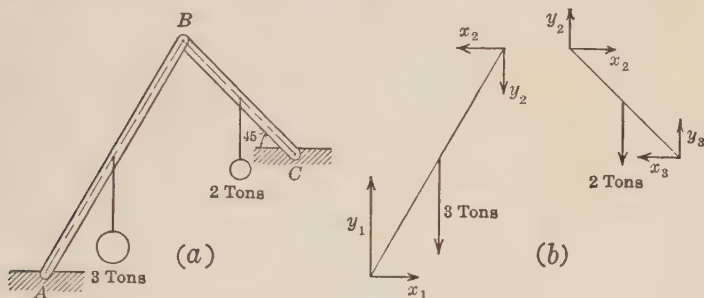


FIG. 68

(SUGGESTION. The pin-force at  $B$  acting on  $AB$  is equal and opposite to that acting on  $BC$  at  $B$ . The forces acting on the separate members are then as shown in Fig. 68 (b). Write the three equations of equilibrium for each of the members and solve the six equations obtained. If the wrong direction has been assumed for  $y_2$ , it will be indicated by a negative value.)

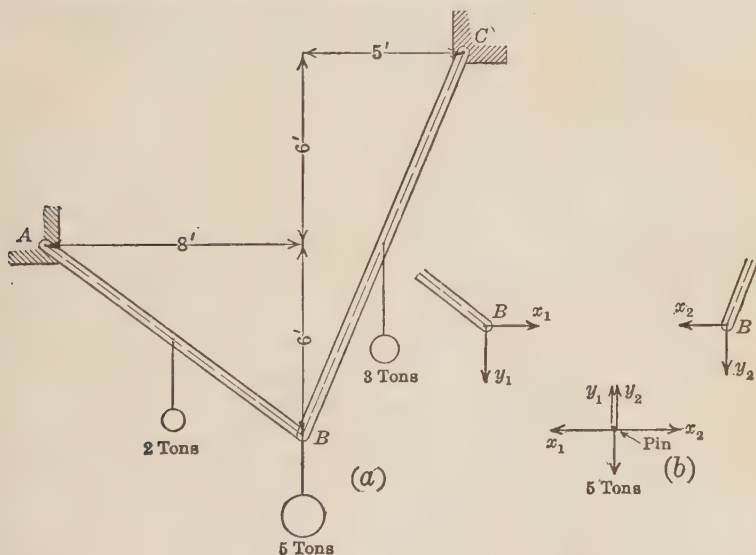


FIG. 69

41. In Fig. 69 (a), the 2-ton and the 3-ton weights are suspended from the centers of  $AB$  and  $BC$  respectively and the 5-ton weight is suspended from the

pin at  $B$ . Compute the horizontal and vertical components of the pin-forces acting on the members at  $A$ ,  $B$ , and  $C$ .

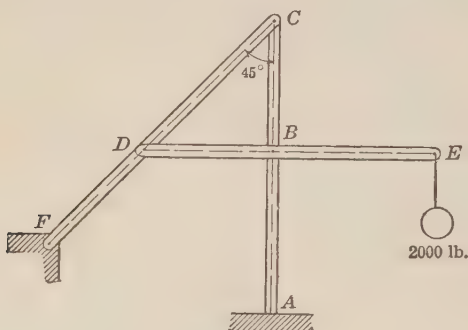


FIG. 70

(SUGGESTION. The forces at  $B$  will act as indicated in Fig. 69 (b), and two equations of equilibrium may be obtained from the forces acting on the pin at  $B$  in addition to the three equations from each of the members  $AB$  and  $BC$ .)

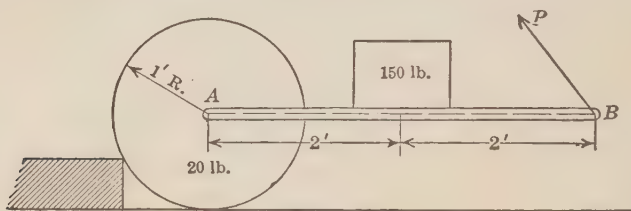


FIG. 71

42. In Fig. 70,  $AB$  is vertical and  $DE$  is horizontal.  $AB = 4$  ft.,  $BC = 3$  ft.,  $BD = 3$  ft.,  $DF = 3$  ft.,  $BE = 4$  ft. The frame rests upon a smooth horizontal floor at  $A$  and is hinged at  $F$ . Find the components of the pin-forces at each pin, and the reaction of the support at  $A$ .

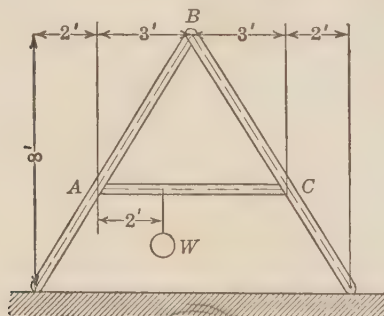


FIG. 72

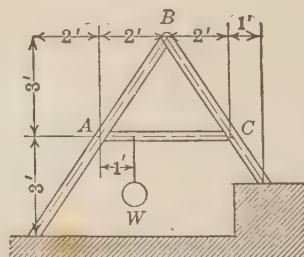


FIG. 73

43. In Fig. 71, find the horizontal and vertical components of the force  $P$  just necessary to start the wheel over the 6-inch step, the bar  $AB$  being horizontal and the wheel weighing 20 lbs.

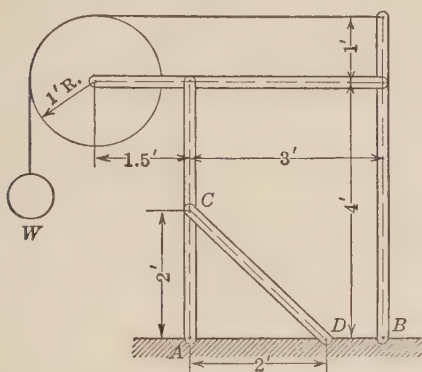


FIG. 74

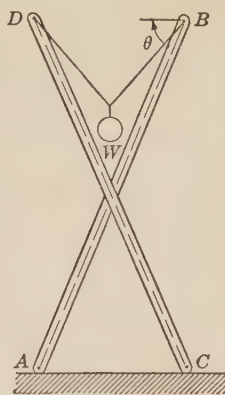


FIG. 75

(Note that the wheel becomes a two-force member under the given conditions).

44. In Fig. 72, the A frame rests upon a smooth horizontal floor; what are the components of the pin forces acting on the members at  $A$ ,  $B$ , and  $C$ ?

45. The frame in Fig. 73 rests on two smooth horizontal supports; what are the components of the pin forces at  $A$ ,  $B$ , and  $C$ ?

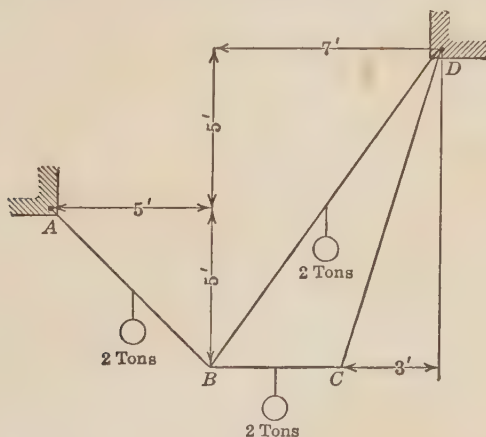


FIG. 76

46. In Fig. 74, find the components of the pin forces at  $A$  and  $B$  and the tension in  $CD$ .

47. In Fig. 75, the bars  $AB$  and  $CD$  are each 20 in. long and are pinned at their centers. A string 12 in. long is attached to  $D$  and  $B$  and a weight  $W$  is

attached to the center point of the string. If the frame is placed on a smooth horizontal floor, show that the angle  $\theta$  which the portions of the string make with the horizontal is  $\cos^{-1} \sqrt{11/27} = 50^\circ 20'$ .

48. In Fig. 76, the weights are suspended from the centers of the members. Neglecting the weights of the members, find the tension in  $CD$  and the components of the pin forces acting on each of the other members.

49. If an additional weight of 2 tons is suspended from the pin at  $B$  in Fig. 76, find the tension in  $CD$  and the components of the pin forces acting on the other members.

**29. Graphical Method of Finding the Resultant of a Set of Coplanar Forces.** Let  $P_1, P_2, P_3, P_4$  be a set of coplanar forces, Fig. 77 (a). It was found in § 26 that the resultant of a set of coplanar forces is a force with components  $\Sigma X$  and  $\Sigma Y$  of the

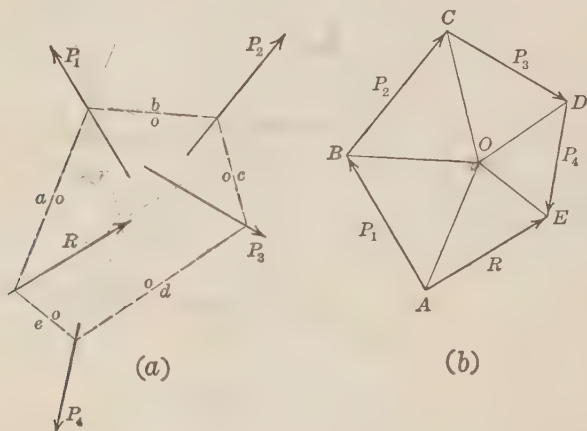


FIG. 77

given forces. The magnitude and direction of this force can be determined by laying off in succession the vectors representing the given forces, as in Fig. 77 (b). The vector  $AE$ , the vector sum of the force vectors, then clearly has components equal to  $\Sigma X$  and  $\Sigma Y$  of the given set of forces, and it therefore represents in magnitude and direction the resultant of the set of forces.

To find the line of action of the resultant, choose any point  $O$  of the plane and connect it with the vertices of the polygon of forces constructed as just described, Fig. 77 (b). Then, choosing any point in the line of any force, as  $P_1$ , in (a), draw through this

point lines parallel respectively to the lines from  $O$  to the ends of the chosen vector,  $P_1$ , in (b). Thus  $a/o$  is parallel to  $OA$ ,  $b/o$  is parallel to  $OB$ . Extend  $b/o$  until it cuts the line of the force that follows  $P_1$  in the polygon of forces,  $P_2$ . From that point draw  $c/o$  parallel to  $OC$ , and so continue to the last force,  $P_4$ , the last line drawn being  $e/o$ . Notice now that in (b) the force  $P_1$  is equivalent to two forces equal in magnitude and direction to  $AO$  and  $OB$ ,  $P_2$  is equivalent to two forces equal in magnitude and direction to  $BO$  and  $OC$ , etc.

Suppose now, in (a), that  $P_1$  be resolved into components along  $a/o$  and  $b/o$ . These components must then be respectively equal in magnitude and in direction to  $AO$  and  $OB$ . In the same way,  $P_2$  may be resolved into components along  $b/o$  and  $c/o$ , equal respectively to  $BO$  and  $OC$ , and the component of  $P_2$  along  $b/o$  then annuls the component of  $P_1$  along that line.

Continuing the construction until all the forces have been resolved into components, there remain only two components, equal to  $AO$  and  $OE$ , along  $a/o$  and  $e/o$ , respectively. The resultant of the set of forces is then simply the resultant of these two components and must therefore pass through their intersection. The magnitude and direction of the resultant having already been determined as those of  $AE$ , the resultant is now known in magnitude, direction, and line of action. The method is clearly applicable to any set of coplanar forces.

The polygon in (b), formed by adding the force vectors, is called the *force polygon*, and the polygon in (a), with sides parallel to the corresponding rays from  $O$  to the vertices of the force polygon, is called the *equilibrium polygon*, or the *funicular polygon*.

Evidently, different choices of  $O$  will give different equilibrium polygons, but not, of course, different lines of action for the resultant, since there is but one resultant.

**30. Graphical Conditions for Equilibrium of a Set of Coplanar Forces.** If in addition to the given forces  $P_1, P_2, P_3, P_4$ , of § 29, there were also a force  $P_5$ , equal but opposite in direction to  $R$ , and acting in the line of  $R$ , the set of forces would be in

equilibrium. The graphical conditions for equilibrium of a set of coplanar forces are therefore that the force polygon shall be a closed polygon, and that, when the equilibrium polygon is constructed, the intersection of the first and last sides shall fall upon the line of the remaining force.

In other words, *the conditions of equilibrium of a set of coplanar forces is that the force polygon shall close and the equilibrium polygon shall close.*

**31. Illustration.** In Fig. 78 is shown a Fink truss carrying loads  $P_1$  and  $P_2$ . The truss is hinged at  $M$  and rests on a roller at  $N$ , so that the reaction at  $N$  is vertical. It is required to find the

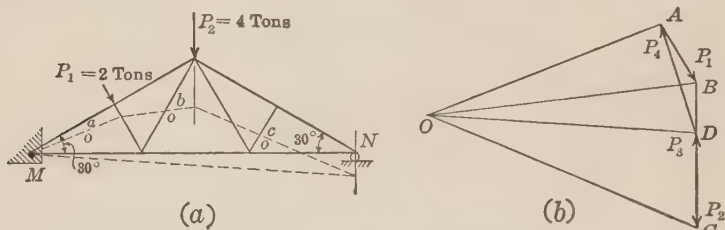


FIG. 78

reactions of the supports by graphical construction. Let the reactions at  $N$  and  $M$  be  $P_3$  and  $P_4$ , respectively. The four forces  $P_1$ ,  $P_2$ ,  $P_3$ ,  $P_4$ , are then in equilibrium. Applying the condition that the force polygon must close, the forces are laid off as far as known in Fig. 78 (b). Starting at  $A$ ,  $P_1$  and  $P_2$  are laid off in succession. Since  $P_3$  is vertical it may next be laid off on the line of  $P_2$ , but of unknown length. The remaining vector,  $P_4$ , must extend from the end of  $P_3$  to  $A$ . To find the vertex of the force polygon where  $P_3$  and  $P_4$  join, construct the equilibrium polygon. Since  $M$  is the only known point on  $P_4$ , the equilibrium polygon should be started at  $M$ , drawing  $o/a$  parallel to  $OA$  from  $M$  to the line of  $P_1$ ; then  $o/b$  parallel to  $OB$  to the line of  $P_2$ ; then  $o/c$  parallel to  $OC$  to the line of  $P_3$ . The remaining side of the equilibrium polygon being drawn, the line  $OD$ , parallel to it, determines the point  $D$  which marks the end of the  $P_3$  and the beginning of  $P_4$ . The vectors  $CD$  and  $DA$  are then scaled for the values of the reactions.



PROBLEMS

50. Make the construction for determining the reactions of the supports of the truss under the loads shown in Fig. 78 and scale the vectors obtained. Also compute the reactions by algebraic methods and compare the results.

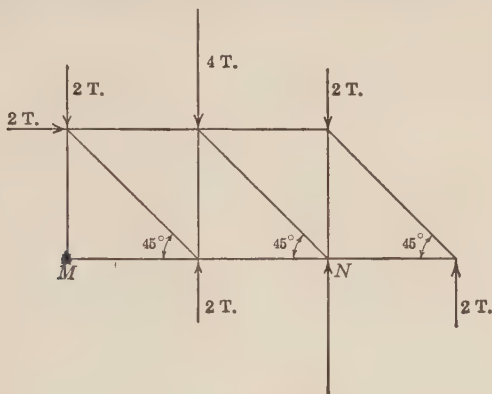


FIG. 79

51. The truss of Fig. 79 is supported at  $M$  and  $N$ , the reaction at  $N$  being vertical. Find the reactions by graphical construction. Also compute the reactions and compare the results.

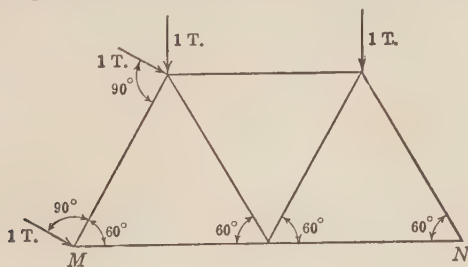


FIG. 80

52. The truss of Fig. 80 is supported at  $M$  and  $N$ . Find the reactions by graphical construction on the supposition: (a) the reactions act in parallel lines; (b) the reaction at  $N$  is vertical, (c) the reaction at  $M$  is vertical.

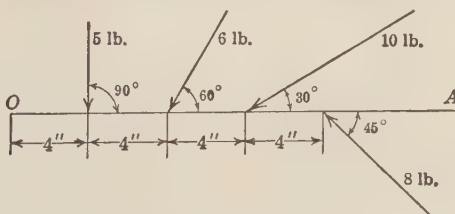


FIG. 81

(NOTE. The work may be somewhat shortened by replacing the two vertical forces by their resultant, and the inclined forces by their resultant.)

53. Find by graphical construction the resultant of the forces shown in Fig. 81. How far from  $O$  does the line of the resultant cut the line  $OA$ ?

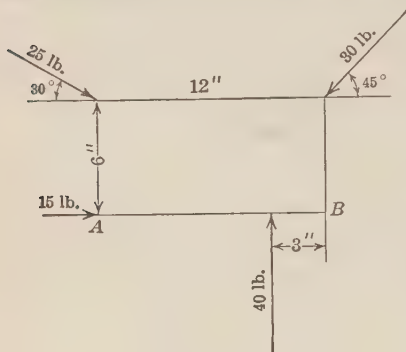


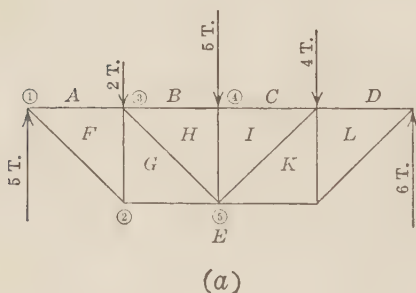
FIG. 82

54. Find by graphical construction the resultant of the four forces shown in Fig. 82, measure the angle which it makes with  $AB$  and the distance from  $A$  to the point of intersection of the resultant and  $AB$ . Solve also by algebraic methods.

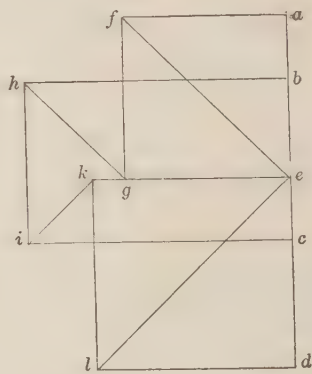
### 32. Graphical Method of Finding Stresses in Trusses.

**Bow's Notation.** Consider the truss of Fig. 83 (a), carrying the loads as shown

and supported at the ends. The acute angles are each  $45^\circ$ , and the reactions of the supports are found by moments to be as given. The joints are all assumed to be pin-joints and the weights of the members are assumed to be included in the given loads at the joints, the members then being regarded as two-force members.



(a)



(b)

FIG. 83

At each joint of the truss, then, a certain set of forces act, and these forces are in the lines of the external forces at the joint and the members of the truss meeting at the joint. The set of forces

at each joint must be in equilibrium and hence must form a closed polygon when added vectorially.

In Fig. 83 (b) are shown the force polygons for all the joints of the truss. Since each member exerts two equal and opposite forces, one at each end of the member, the force polygon at any point will have two or more sides equal to the sides of a force polygon at some other point. Considerable labor is therefore saved by so constructing the force polygon that one side will serve to represent the force at two points.

The construction of Fig. 83 (b) is made as follows. The external forces of the truss, consisting of the loads and the reactions of the supports, form a set of forces in equilibrium and therefore form a closed polygon when laid off in succession. This is done by starting with any force and adding vectorially the remaining forces in the order in which they are met in passing around the truss. The regions between the forces and between the members are marked by letters  $A, B, C$ , etc., and any force or any member is indicated by naming the two letters in the regions which that force or that member separate. Thus  $AB$  means the 2-ton force between the regions  $A$  and  $B$ ,  $EA$  the left reaction,  $AF$  the left horizontal member, etc. In laying off the polygon of external forces, called the *load line*, the small letters are used to represent the forces between the regions marked by the corresponding large letters. Thus  $ab$  represents the force between  $A$  and  $B$ ,  $bc$  the force between  $B$  and  $C$ , etc. Having completed the load line  $abcdea$ , a point of the truss is selected at which only two unknown forces act, such as the left support, marked (1).

Using the known force  $ea$  of the load line, lines are drawn through its ends parallel to the other forces acting at that point, through  $e$ , a line parallel to  $EF$  and through  $a$ , a line parallel to  $AF$ . Calling the intersection of these lines  $f$ , the triangle of forces  $aef$  is obtained, the sides of which represent the forces acting between the corresponding regions around the point (1). Thus  $af$  represents the force in the member  $AF$ ,  $fe$  the force in  $FE$ . Also the directions of the forces acting at (1) are determined by the fact that in the force polygon for any point the vector sum of the force vectors at that point is zero (§ 11). Thus since  $ea$  is an

upward force, the vector connecting  $a$  and  $f$  begins at  $a$  and ends at  $f$ , and the vector between  $f$  and  $e$  points from  $f$  to  $e$ . This means that the force in the member  $AF$  acts toward the point (1) and the force in the member  $FE$  acts away from the point (1). Therefore  $AF$  is in compression and  $FE$  is in tension.

The forces in these members may now be regarded as known forces acting at points (3) and (2) in opposite directions from their actions at (1). The remaining forces at (2) may now be found by completing the triangle of which  $ef$  is one side and the other sides are parallel, respectively, to  $FG$  and  $GE$ . This is done by drawing through  $f$  a line parallel to  $FG$  and through  $e$  a line parallel to  $EG$ . Calling their intersection  $g$ , the triangle of forces  $efg$  is obtained for point (2), in which  $fg$  represents the force (compression) in  $FG$ , and  $ge$  the force (tension) in  $GE$ . Knowing now the forces in  $AF$  and  $FG$ , there remain only two unknown forces at point (3), namely, those in  $BH$  and  $HG$ . These may now be obtained by completing the polygon  $gfab \dots$ , by drawing through  $g$  a line parallel to  $GH$  and through  $b$  a line parallel to  $BH$ . Calling their intersection  $h$ , the polygon  $gfabh$  is the force polygon for point (3), the new forces determined being  $bh$ , representing the force (compression) in  $BH$ , and  $hg$ , representing the force (tension) in  $HG$ . Next the force polygons

for the points (4), (5), etc. may be constructed in order, obtaining the complete figure.

The figure containing the force polygons of all the joints of the truss, figure (b) of Fig. 83, is called the **stress diagram** of the truss.

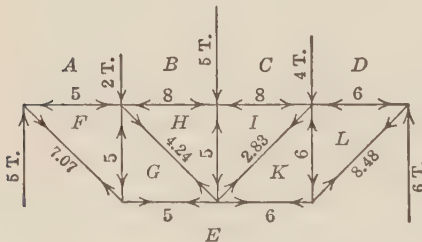


FIG. 84

The notation used, in which the large letters mark the regions bounded by the lines of the forces and the small letters the vertices of the stress diagram, is known as **Bow's notation**.

In Fig. 84, a sketch of the truss is shown with the values of the forces (stresses) written along the various members, and arrows

on the members indicating whether the members are in tension or compression. Arrows on any member pointing toward its ends indicate compression in that member, and arrows pointing away from the ends indicate tension in the member, since the arrows indicate the forces acting upon the pins. Thus  $AF$  is in compression,  $EF$  in tension. Another method of indicating the kind of stress is to indicate tension by a plus sign and compression by a minus sign.

**33. Stress Diagram for Inclined Loads.** The method used in § 32 is immediately applicable to trusses carrying inclined loads or to a combination of vertical and inclined loads in the same plane.

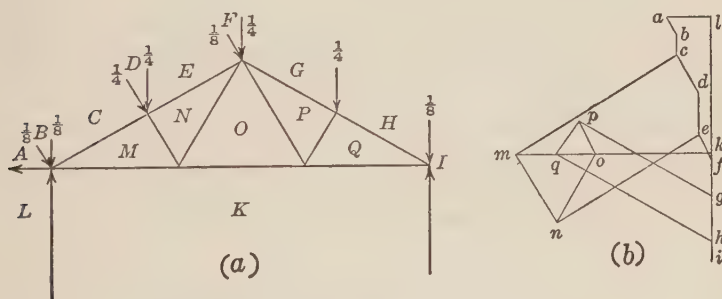


FIG. 85

Figures 85 (a) and 85 (b) show, respectively, a truss carrying both vertical and inclined loads, and the corresponding stress diagram. As in the former illustration, the load line is first laid off, taking the forces as they are met in passing around the outside of the truss. It was assumed that the right reaction was vertical and the reactions were found by the application of the algebraic conditions of equilibrium. Having laid off the load line  $abcdefghikla$ , the point (1) is selected as having only two unknown forces and lines are drawn through  $c$  parallel to  $CM$  and through  $k$  parallel to  $MK$ , determining the point  $m$  and thus completing the force polygon for the point (1). Next a line is drawn through  $e$  parallel to  $EN$  and through  $m$  parallel to  $MN$ , determining  $n$  and completing the force polygon for the point (2), etc.



**34. The Method of Sections.** In finding the stresses in the members of a pin-connected truss, it is often convenient to use the method of sections. As an illustration of this method consider again the truss of Fig. 83. We may think of the truss as divided

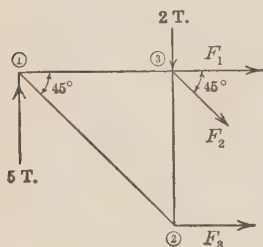


FIG. 86

with the other to be removed and replaced by the forces which these members exert on the pins at their ends. If the members  $BH$ ,  $HG$ , and  $GE$  were thus replaced by the forces which they exert, the left portion of the truss so divided would be in equilibrium under the action of forces, as shown in Fig. 86. The algebraic conditions of equilibrium may then

be applied to this part of the truss and the following equations written, the last equation being obtained by taking moments about the point (3),

$$\Sigma X = F_3 + F_2 \cos 45^\circ + F_1 = 0,$$

$$\Sigma Y = 5 - 2 - F_2 \sin 45^\circ = 0,$$

$$\Sigma Pd = F_3 - 5 = 0,$$

from which are obtained  $F_3 = 5$ ,  $F_2 = 4.24$ ,  $F_1 = -8$ .

Since there are only three independent equations of equilibrium, the stresses in not more than three members can be found by any division of the truss into two parts. Any section dividing the truss into two parts must therefore not cut across more than three members.

#### PROBLEMS

55. The angles of the pin connected truss of Fig. 87 are each  $60^\circ$ . Find the reactions of the supports, draw the stress diagram and scale the stresses. Check the results in three members by the method of sections. Make a sketch of the truss and write the values of the stresses along the corresponding members, using a plus sign to indicate tension and a minus sign to indicate compression.

(56) Construct the stress diagram for the truss of Fig. 79, scale the stresses and write their values on a sketch of the truss, indicating tension by a plus sign and compression by a minus sign. Check the stresses in three

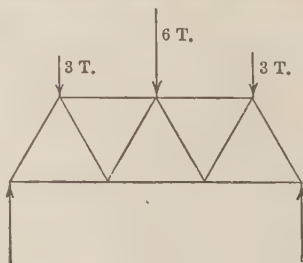


FIG. 87



members by the method of sections. What percent of error do you find in your results obtained graphically?

57. Construct the stress diagram for the truss of Fig. 80, and tabulate the results. Check by the method of sections.

58. Construct the stress diagram and scale the stresses for the truss of Fig. 88, carrying the loads as shown. Check by finding the stresses in three members by the method of sections.

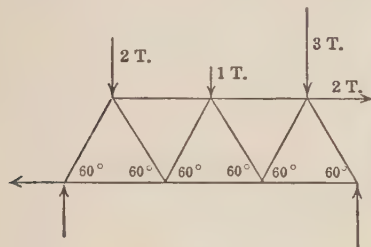


FIG. 88

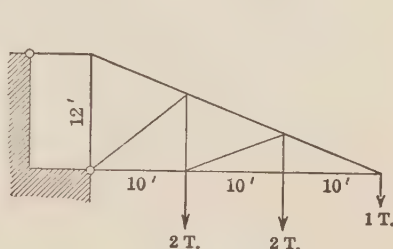


FIG. 89

59. Construct the stress diagram for the cantilever truss of Fig. 89, the upper reaction being horizontal. Draw the stress diagram without first finding the reactions of the supports. The reactions will be determined as part of the stress diagram. Scale the stresses and write the values on a sketch of the truss.

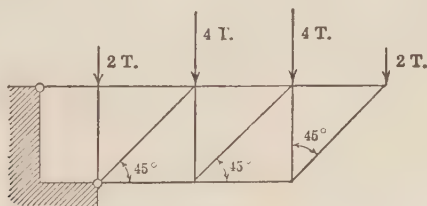


FIG. 90

60. Without first finding the reactions of the supports, construct the stress diagram and scale the stresses for the truss of Fig. 90. Check by finding the stresses in three members by the method of sections.

61. In Fig. 85 assume the total vertical load to be unity and to be equally distributed on the four panels as shown, half of the load on each panel falling at the end of the panel in each case. Assume that the wind load, perpendicular to the panels, is one-half that of the total vertical load, and is also equally distributed on the two left panels as shown. (The possible wind force is resolved into components perpendicular to and parallel to the roof carried by the truss, and the component parallel to the roof is neglected as not actually acting on the roof.) Compute the reactions algebraically, assuming that the right reaction is vertical; lay off the load line and construct the stress diagram. Scale the stresses and tabulate on a sketch of the truss. What graphical

check do you have on the accuracy of your work? Check the stresses in  $EN$ ,  $NO$ , and  $OK$  by the method of sections.

62. Construct the stress diagram for the truss of Fig. 91, scale the values of the stresses, and write their values on a sketch of the truss, indicating tension and compression by plus and minus signs.

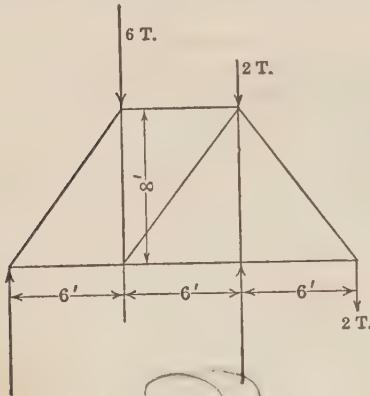


FIG. 91

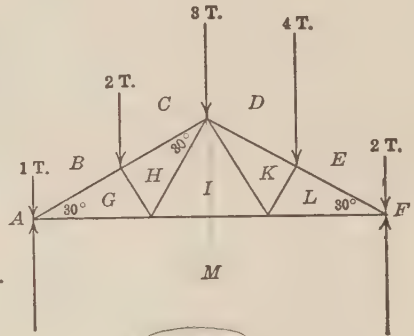


FIG. 92

63. Construct the stress diagram for the Fink truss of Fig. 92, scale the stresses and write the values of the stresses on a sketch of the truss, indicating tension and compression by plus and minus signs. Check the stresses in  $CH$ ,  $HI$ , and  $IM$  by the method of sections.

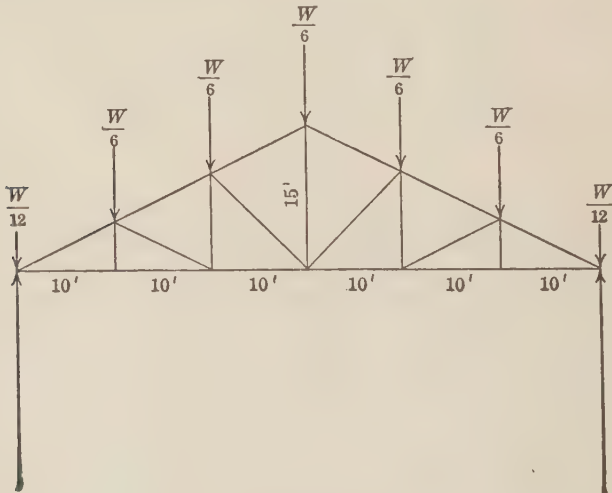


FIG. 93

64. In the Howe truss of Fig. 93, let the total load be  $W$ , equally distributed on the panels as shown. Construct the stress diagram and scale the stresses.

65. Using the truss of Fig. 93, assume a wind load only to act on the left side of the truss, of value  $W'$ , equally distributed on the panels and perpendicular to the panels. Assume the reactions at the supports to be parallel. Find the reactions graphically and check by computation. Construct the stress diagram and scale the stresses.

66. Using the truss of Fig. 93, assume that the vertical load of Problem 64 and the wind load of Problem 65 act simultaneously, and that the reactions of the supports are in parallel lines. Let  $W = 12$  tons,  $W' = 6$  tons. Find the reactions graphically and check by computation. Construct the stress diagram and scale the stresses. Compare the stresses in the various members with the algebraic sum of the stresses found in the two preceding problems for the same members when  $W$  and  $W'$  have the same values as are here assigned.

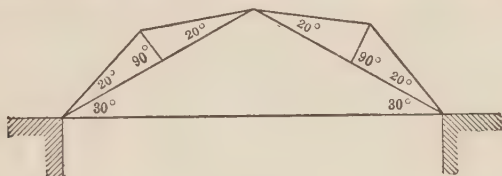


FIG. 94

67. In the truss of Fig. 94, assume that the wind load on the lower right panel is 2 tons, normal to the panel and applied equally at the ends of the panel, and that the wind load on the second panel on the right is  $1/2$  ton, normal to the panel and applied equally at its ends. Assume the reactions of the supports to be in parallel lines. Find the reactions, draw the stress diagram and scale the stresses due to the wind load alone.

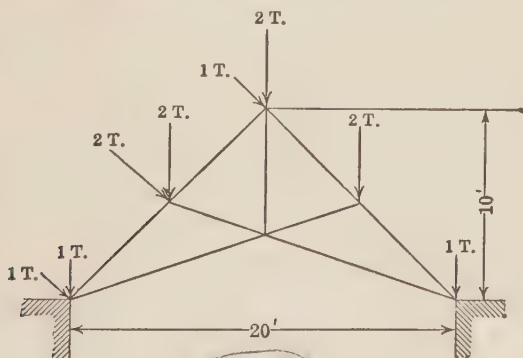


FIG. 95

68. Assume a total vertical load of 8 tons on the truss of Fig. 94 and construct the stress diagram, assuming the loads to be equally distributed over the panels and applied at the joints. Scale the stresses.

69. Assume the loads of the two preceding problems to act simultaneously on the truss of Fig. 94 and construct the stress diagram, assuming the reactions of the supports to be in parallel lines.

70. In Fig. 95 is shown a sketch of a scissors truss. The four panels are equal. Construct the stress diagram: (a) for the vertical loads alone, (b) for the wind load alone, assuming parallel reactions of the supports, (c) for both loads acting together, assuming the reactions of the supports to be in parallel lines. In each case scale the stresses.

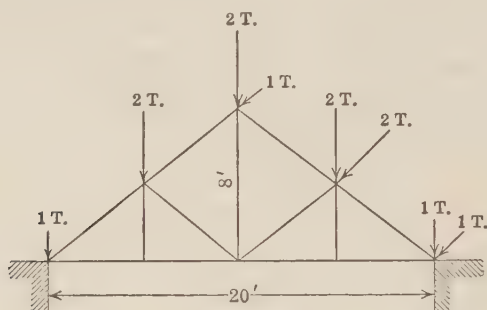


FIG. 96

71. Fig. 96 represents a king post truss. Construct the stress diagram: (a) for the vertical load alone, (b) for the wind load alone, (c) for the combined loads, assuming parallel reactions of the supports. In each case scale the stresses.

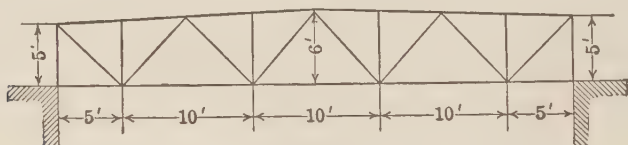


FIG. 97

72. Fig. 97 represents a Warren truss. Construct the stress diagram for a total vertical load  $W$ , equally distributed on the eight equal panels. Scale the stresses and tabulate them on a sketch of the truss. Check the stresses in three members near the center by the method of sections.

**35. The Method of Substitution.** In constructing the stress diagram for some trusses, a stage in the construction may be reached at which it is not possible to find an additional point at which there are only two remaining unknown forces. The usual procedure cannot then be followed, as it is not possible to complete a polygon with three unknown sides in known directions only. For example, for the truss of Fig. 98, having laid off the load line,

$abc \dots fa$ , the force polygons for the points (1), (2) and (3) may be found in succession, but no additional point can be found in the left-half of the truss at which there only two unknown forces acting. Suppose now that the members  $KL$  and  $LM$  be removed and a single member,  $OM$ , be put in, as in Fig. 99. This may be

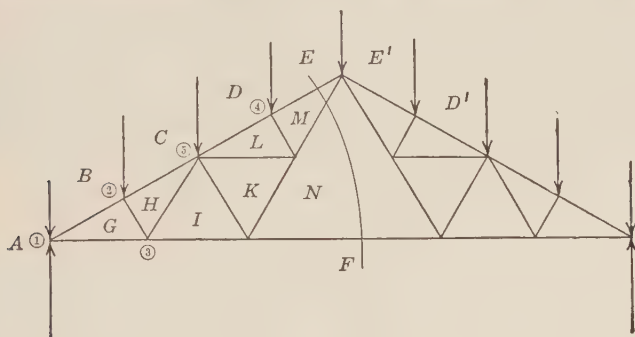


FIG. 98

done without changing the stresses in  $EM$ ,  $MN$ , and  $FN$ , for if a cut be made across these three members in Fig. 98 and Fig. 99, and the part of the truss to the left be considered as a body in equilibrium, there would be in the two cases precisely the same

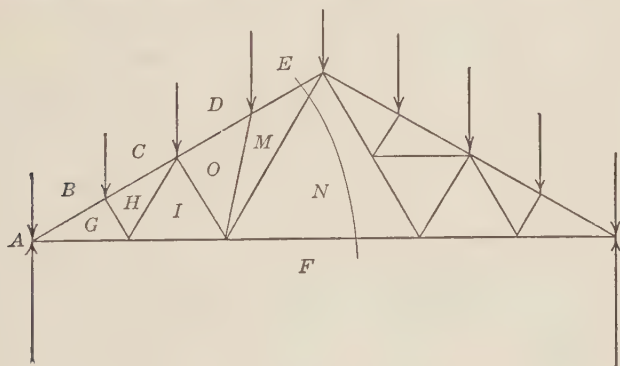


FIG. 99

known forces, and the three unknown forces in the members cut would act in the same lines. The equations for the determination of the three unknown forces in  $EM$ ,  $MN$ , and  $NF$  would be exactly the same in the two cases, and hence the forces themselves

would be the same. Therefore the point designated by  $m$  in the stress diagram for the truss of Fig. 99 would be the same as for the truss of Fig. 98. The stress diagram for Fig. 99 can be constructed in the order of the points marked, thus obtaining the point  $m$  of the diagram at point (5). The construction for points (1), (2), and (3) would be the same for the two figures. Having thus found the point  $m$  by the use of Fig. 99, the polygon  $emld$  of Fig. 98 may then be constructed, determining the stress  $dl$ ; and the polygon  $ihcdlk$  may be constructed, since only the point  $k$  will then be lacking. The completion of the stress diagram will then offer no further difficulty.

### PROBLEMS

73. Assume that the truss of Fig. 98 carries a total vertical load  $W$  equally distributed on the panels. Construct the stress diagram by the method explained above. It is, of course, not necessary to draw a figure of a second truss, but only to locate the point  $m$  in the stress diagram by putting in a dotted line  $OM$  in the original figure of the truss and leave out the lines  $KL$  and  $LM$  in constructing the polygons for the points (5) and (4).

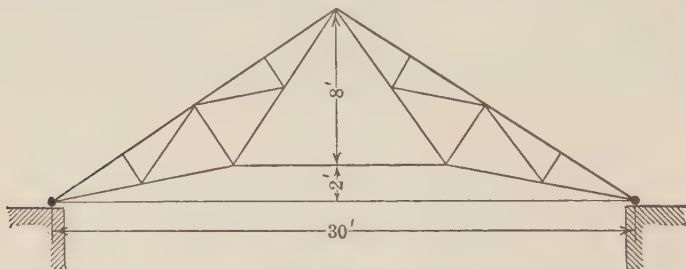


FIG. 100

74. The truss of Fig. 100 carries a vertical load of 12 tons on the left half of the truss, equally distributed on the four equal panels, and a vertical load of 16 tons similarly distributed on the right half. Construct the stress diagram, using the method of substitution. (The substitution is necessary in only one-half of the truss.)

Check the stress in the horizontal member of the lower chord by the method of sections.

75. Figure 101 represents a compound fan truss of pitch  $1/4$ . (The *pitch* is the ratio of height to length of span.) The twelve panels of the upper chord are all equal.  $MN$  is perpendicular to the rafter,  $ON$  is horizontal and joint (4) is midway between joints (1) and (8). Construct the stress diagram for a load  $W$  equally distributed on the twelve panels.



(SUGGESTION. Construct the stress diagram for joints (1), (2), (3), and (4). Remove members  $NO$ ,  $OP$ , and  $PQ$ ; transfer half of the load  $EF$  to point (5) and half to point (7) and substitute temporarily a member between points

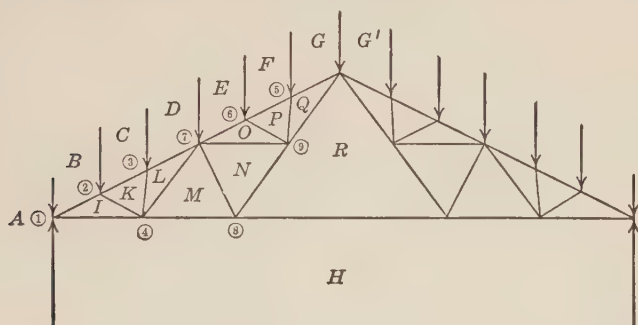


FIG. 101

(5) and (8). Continue the stress diagram for points (7) and (5), thus determining the point  $q$ . Having determined  $q$ , restore the members  $PQ$ ,  $OP$ , and  $ON$ , and continue the stress diagram for points (5), (6), (7), (8), and (9), in that order.

## CHAPTER II

### FORCES IN SPACE

**36. Resolution of a Force into Components.** Since a force admits of vector representation we may, as in § 7, resolve a force into components in space. Thus, the force  $P$  (Fig. 102) may be

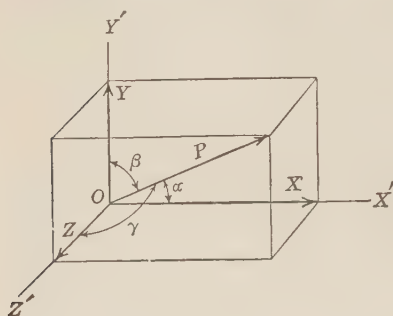


FIG. 102

resolved into the three mutually rectangular components  $X, Y, Z$ , parallel to a set of coordinate axes where

$$X = P \cos \alpha,$$

$$Y = P \cos \beta,$$

$$Z = P \cos \gamma,$$

and where

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

The angles  $\alpha, \beta, \gamma$  which the vector makes with the axes of coordinates are called the *direction angles* of the vector and their cosines the *direction cosines* of the vector.

**37. Resultant of a Set of Concurrent Forces in Space. Equilibrium.** If several forces act at the same point, each can be resolved into components  $X, Y, Z$  along rectangular axes through the point; the components along the axes can then be combined into  $\Sigma X, \Sigma Y, \Sigma Z$  and these again combined into a single force,

$$R = [(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2]^{1/2},$$

acting through the given point in a direction whose direction cosines are

$$\cos \alpha = \frac{\Sigma X}{R}, \quad \cos \beta = \frac{\Sigma Y}{R}, \quad \cos \gamma = \frac{\Sigma Z}{R}.$$

If the forces are in equilibrium, each of the three resultant components must vanish, and conversely, if each of the three

resultant components vanishes, the forces are in equilibrium. Hence necessary and sufficient conditions that a set of concurrent forces in space be in equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

**38. Combination of Couples in Intersecting Planes.** Let the two intersecting planes  $AC$  and  $AD$ , Fig. 103, each contain a couple. Wherever the couples may be situated in these planes, they may be replaced by any couples in these same planes which have equal moments respectively, by § 22. Let the moment of the couple in the plane  $AC$  be  $M_1$  and that of the couple in  $AD$  be  $M_2$ . Replace these couples by respectively equal couples in their own planes, having a common arm  $a$  along the

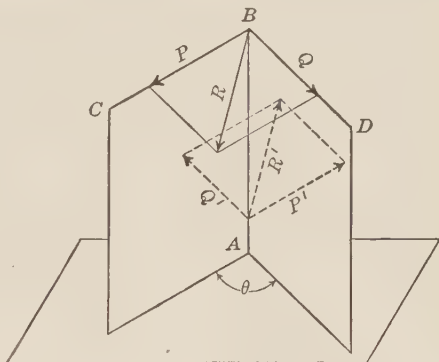


FIG. 103

line of intersection of the two planes, and let the forces of the couples be laid off from that line. If  $(P, P')$  are the forces of one couple and  $(Q, Q')$  those of the other, then  $M_1 = Pa$  and  $M_2 = Qa$ . The two forces  $P$  and  $Q$  and the two forces  $P'$  and  $Q'$  may be combined respectively into the equal and opposite forces  $R$  and  $R'$ , which therefore form a couple. The moment of this couple is  $M = Ra$ .

If  $\theta$  is the angle between the planes  $AC$  and  $AD$ , then

$$(1) \quad R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

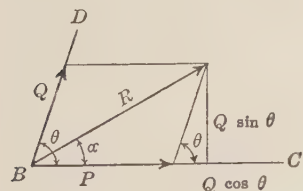


FIG. 104

If  $\alpha$  is the angle which the plane of the resultant couple makes with the plane of the couple  $(P, P')$ , then (see Fig. 104) we have

$$(2) \quad \tan \alpha = \frac{Q \sin \theta}{P + Q \cos \theta}.$$

Multiplying both sides of equation (1) by  $a^2$ , and the numerator and denominator of equation (2) by  $a$ , we may write these equations in the form

$$(3) \quad M^2 = M_1^2 + M_2^2 + 2 M_1 M_2 \cos \theta,$$

$$(4) \quad \tan \alpha = \frac{M_2 \sin \theta}{M_1 + M_2 \cos \theta}.$$

**39. Couples in Parallel Planes.** In Fig. 105 are represented six numerically equal forces acting along four parallel edges of a rectangular parallelepiped. Since the four forces in the lower plane annul each other, the whole set of forces reduces to a couple in the upper plane with moment  $aP$ . But regarded otherwise, the forces  $P$  and  $P_1'$  may be combined into a resultant of value  $2P$ , acting at  $O$ , the mid-point of  $BC'$ ; and the forces  $P_1$  and  $P'$  may be combined into a force of value  $2P$ , also acting at  $O$ , but in the opposite direction from the first resultant and therefore annulling it.

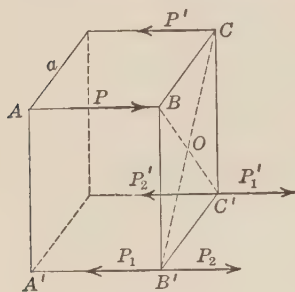


FIG. 105

The set of forces are therefore equivalent to the forces  $P_2$  and  $P_2'$ , which form a couple of moment  $aP$  in the lower plane. Since the same set of forces are equivalent to the couple in the upper plane or to the couple in the lower plane, the two couples are equivalent. Since also either couple may be replaced by any couple of equal moment in its own plane (§ 22), it follows that *any two couples with equal moments, acting in parallel planes, are equivalent.*

**40. Vector Representation of Couples.** The results established in § 38, namely, equations (3) and (4), suggest the possible representation of couples as vectors, since the formulas connecting the moments of the original couples and the resultant couple have the same form as those obtained in combining vectors by the cosine law. It is thus found that the results of § 38 and also of §§ 21, 22, and 39 are in agreement with the following repre-

sentation: A couple may be adequately represented by a vector of length equal to the moment of the couple, laid off in any line perpendicular to the plane of the couple, and pointing in the direction that a right hand screw would advance when turned in the direction indicated by the forces of the couple.

The vector is directed at right angles to the plane because that is the only direction that bears the same relation to all couples in the same plane and it is made equal to the moment of the couple so that a single vector may represent any one of a set of couples of equal moments acting in the same plane or in parallel planes; for all such couples are equivalent, by §§ 22 and 39, and should therefore have the same representation. With this vector representation, the results of combining vectors in the same plane or in intersecting planes agrees with those already found in §§ 21 and 38, the resulting vector in each case being perpendicular to the plane in which the resultant couple acts. This vector representation of couples therefore gives in each case the same results as are obtained by regarding the couples as made up of forces and applying to them the laws of forces. It must be noted, however, that vectors representing couples and those representing forces must be kept separate. There can be no combination of one with the other. They can be combined only when the couple vector is replaced by the two force vectors which the couple vector represents. It is to be noted also that the line of action of a couple vector is of no consequence; only its direction and magnitude matter, whereas a force vector is not completely determined until its line of action is known.

**41. Illustration.** In Fig. 106 (a) are represented three couples of 45 lb.-in., 60 lb.-in. and 40 lb.-in. in the  $xy$ ,  $yz$ , and  $zx$  planes, respectively; and in Fig. 106 (b) are shown vectors representing these couples and the resultant vector of moment  $M$ . The value of  $M$  is  $[(40)^2 + (45)^2 + (60)^2]^{1/2} = 85$  lb.-in., and the angles which the resultant vector makes with the axes are given by the equations

$$\cos \lambda = 60/85, \quad \cos \mu = 40/85, \quad \cos \nu = -45/85,$$

from which  $\lambda = 45^\circ 6'$ ,  $\mu = 61^\circ 56'$ ,  $\nu = 121^\circ 58'$ . Hence the

resultant of the three given couples is a couple of moment 85 lb.-in. in any plane perpendicular to the line whose direction angles are  $54^\circ 6'$ ,  $61^\circ 56'$ ,  $121^\circ 58'$ , the direction of the forces forming the couple being such as would cause right-handed screw motion in the direction of the vector  $M$  in Fig. 106 (b).

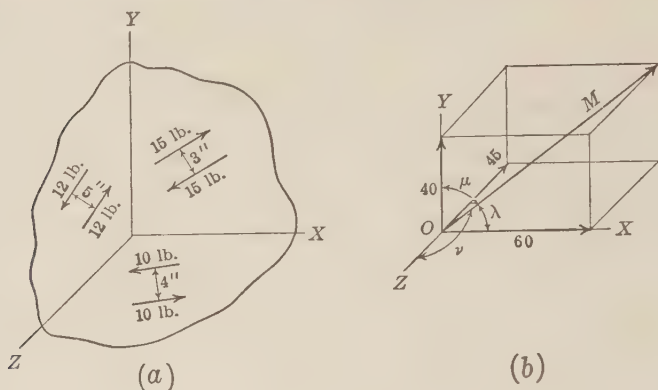


FIG. 106

## PROBLEMS

76. Find the value of the moment of the resultant couple and the direction angles of the normal to the plane in which the resultant couple acts, of the three couples of Fig. 107.

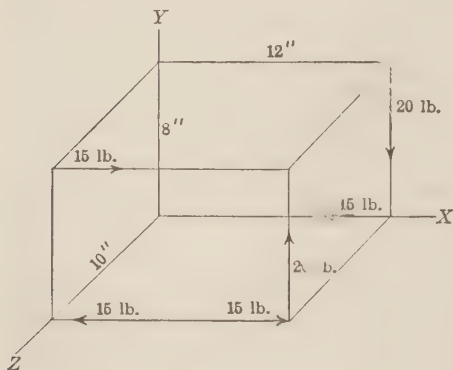


FIG. 107

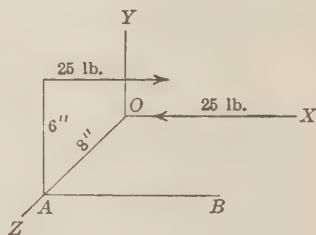


FIG. 108

77. A couple of moment 60 lb.-in. is represented by a vector with direction cosines  $1/4$ ,  $1/3$ , and  $+$  ( ). Resolve the couple into component couples in the coordinate planes. Show the vector representation on a sketch.



78. Resolve the couple of Fig. 108 into two component couples in coordinate planes.

(SUGGESTION. Put in two equal and opposite forces in the line  $AB$ .)

79. Replace the three forces of Fig. 109 by a single force acting at  $O$  and a single couple. Find the magnitude and direction angles of the resultant force vector and of the resultant couple vector. Make separate sketches for the force vectors and the couple vectors.

(SUGGESTION. Replace each force by a force at  $O$  and a couple and then combine couples and forces separately.)

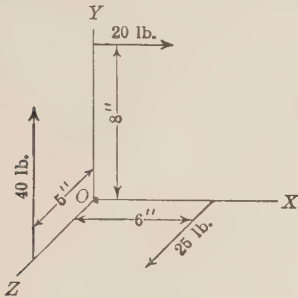


FIG. 109

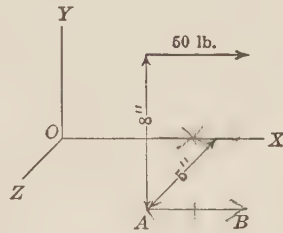


FIG. 110

80. Replace the force of 50 lbs. parallel to the  $x$  axis in Fig. 110 by a force at  $O$ , a couple in the  $xy$  plane and a couple in the  $xz$  plane.

(SUGGESTION. Put in equal and opposite forces in the line  $AB$  and also in the  $x$  axis.)

81. In Fig. 111, all the forces are parallel to the  $x$  axis and are in the coordinate planes indicated. Find a single force that will balance the five forces.

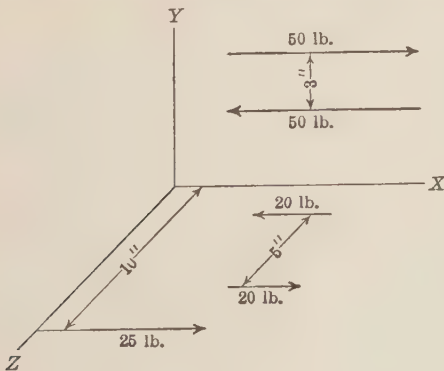


FIG. 111

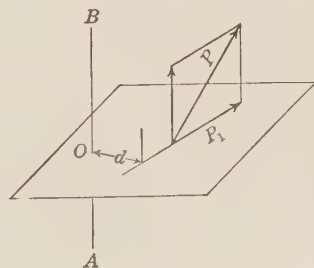


FIG. 112

42. **Moment of a Force with Respect to a Line.** The *moment* of a force with respect to a line is defined to be the moment of the projection of the force upon a plane perpendicular to the line

1 Come

with respect to the intersection of the plane and the line. Thus, in Fig. 112, the force  $P$  is resolved into components, one of which is parallel to the line  $AB$  and the other,  $P_1$ , is in a plane perpendicular to  $AB$  at a point  $O$ . The moment of  $P$  with respect to  $AB$  is then defined as the moment of  $P_1$  with respect to  $O$ , that is,  $P_1d$ . No sign will be attached to this product at present except to agree that moments in opposite directions will be of different signs. Evidently, from the definition given, it makes no difference in the value of the moment of a force with respect to a line if the force vector be moved in its line of action, or if the plane be moved to any parallel position.

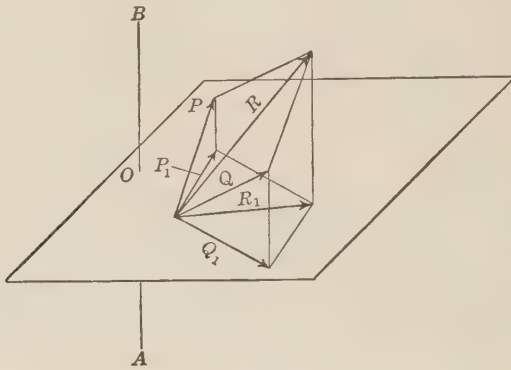


FIG. 113

**43. Moment of the Resultant of Two Intersecting Forces with Respect to a Line.** In Fig. 113,  $P$  and  $Q$  are two intersecting forces,  $R$  is their resultant, and  $AB$  is any line. Through the intersection of the forces pass a plane perpendicular to  $AB$  and let  $P_1$ ,  $Q_1$ , and  $R_1$  be the projections of  $P$ ,  $Q$ , and  $R$  respectively on this plane. The parallel and equal sides of the parallelogram formed on  $P$  and  $Q$  have parallel and equal projections and therefore  $R_1$  is the diagonal of a parallelogram formed on  $P_1$  and  $Q_1$  and hence is the resultant of the components  $P_1$  and  $Q_1$ . By § 17, the moment of  $R_1$  is equal to the sum of the moments of  $P_1$  and  $Q_1$  with respect to  $O$ , and since the moments of the forces with respect to  $AB$  are equal to the moments of their projections with respect to  $O$ , it follows that the sum of the moments of  $P$  and

$Q$  with respect to  $AB$  is equal to the moment of  $R$  with respect to  $AB$ . In other words, *the sum of the moments of two intersecting forces with respect to any line is equal to the moment of their resultant with respect to that line.*

**44. Moment of a Force with Respect to Coordinate Axes.** In Fig. 114 is shown a force  $P$  with components  $X$ ,  $Y$ ,  $Z$  and point of application  $A(x, y, z)$ . The moment of  $P$  with respect to the  $x$  axis is the moment of the projection,  $AB$ , on a plane through  $A$  perpendicular to the  $x$  axis, with respect to the point  $C$  where this plane cuts the  $x$  axis, and the moment of  $AB$  with respect to  $C$  is in turn equal to the sum of the moments of its components  $Y$  and  $Z$  with respect to  $C$  (§ 17). Adopting the convention that positive moments are such as would tend to cause rotation from  $+x$  to  $+y$ , from  $+y$  to  $+z$ , and from  $+z$  to  $+x$ , the moment of  $P$  with respect to the  $x$  axis is  $yZ - zY$ .

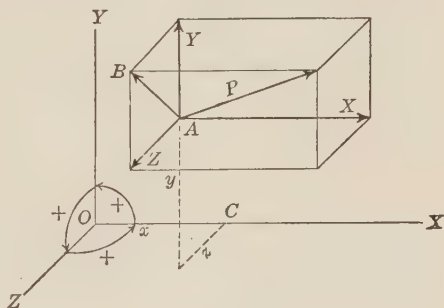


FIG. 114

It follows therefore that for any number of forces the sum of the moments of all the forces with respect to the  $x$ ,  $y$ , and  $z$  axes respectively are

$$\begin{aligned} \Sigma(yZ - zY), \quad \Sigma(zX - xZ), \\ \Sigma(xY - yX). \end{aligned}$$

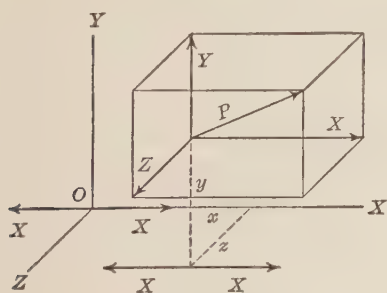


FIG. 115

**45. Resultant of a Set of Forces in Space.** Let  $P$  be any force in space with point of application at  $(x, y, z)$  and with components  $X$ ,  $Y$ ,  $Z$  parallel to the coordinate axes, Fig. 115. By putting in two pairs of equal and opposite forces  $X$  in the lines shown

$Z$  parallel to the coordinate axes, Fig. 115. By putting in two pairs of equal and opposite forces  $X$  in the lines shown

in the figure, the component  $X$  of the force  $P$  is seen to be equivalent to a force  $X$  acting along the  $x$  axis, and two couples of moments  $zX$  and  $-yX$  in the  $xz$  plane and the  $xy$  plane, respectively. In the same way, the component  $Y$  may be shown to be equivalent to a force  $Y$  acting in the  $y$  axis and two couples of moments  $xY$  and  $-zY$  in the  $xy$  plane and the  $yz$  plane respectively, and the component  $Z$  is equivalent to a force  $Z$  acting in the  $z$  axis and two couples of moments  $yZ$  and  $-xZ$  in the  $yz$  plane and the  $xz$  plane respectively. Hence the force  $P$  acting at  $(x, y, z)$  is equivalent to a force equal and parallel to the original force, but acting at the origin, and three couples of moments  $xY - yX$ ,  $yZ - zY$ , and  $zX - xZ$  in the  $xy$ ,  $yz$ , and  $xz$  planes, respectively.

Any number of forces applied at points  $(x, y, z)$  and with components  $X, Y, Z$  parallel to the axes are therefore equivalent to three forces  $\Sigma X, \Sigma Y, \Sigma Z$  acting at the origin along the  $x, y, z$  axes, respectively, and three couples of moments  $\Sigma(xY - yX)$ ,  $\Sigma(yZ - zY)$ ,  $\Sigma(zX - xZ)$ , acting in the  $xy, yz$ , and  $xz$  planes respectively. Letting

$$M_x = \Sigma(yZ - zY), \quad M_y = \Sigma(zX - xZ), \quad M_z = \Sigma(xY - yX),$$

we may combine the forces into a single force

$$R = [(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2]^{1/2},$$

acting at the origin, and the couples into a single couple of moment

$$M = [M_x^2 + M_y^2 + M_z^2]^{1/2}.$$

The direction cosines of the force vector are

$$\Sigma X/R, \quad \Sigma Y/R, \quad \Sigma Z/R,$$

and the direction cosines of the normal to the plane containing the resultant couple are

$$M_x/M, \quad M_y/M, \quad M_z/M.$$

Comparing the values of  $M_x, M_y, M_z$  with the values of the sums of the moments of the forces about the  $x, y, z$  axes found in § 44, it is seen that they are respectively equal.

**46. Conditions of Equilibrium of a Set of Forces in Space.** If the forces of § 45 are in equilibrium the resultant force and the resultant couple must both vanish, since a force and a couple cannot balance each other (§ 23). The conditions for the vanishing of the resultant force are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

and for the vanishing of the resultant couple are

$$M_x = 0, \quad M_y = 0, \quad M_z = 0.$$

Hence a set of forces in space are in equilibrium when, and only when, the sums of the components of the forces parallel to three rectangular axes are zero, and the sums of the moments of the forces about these axes are zero.

**47. Simplification of a Force and a Couple.** It was shown in § 45 that any set of forces may be reduced to a single force and a single couple. A simplification of this result may be made as follows. In Fig. 116 is shown a force

$P$  making an angle  $\theta$  with a plane containing a couple of moment  $M$ . This couple may be represented by a vector perpendicular to the plane and then resolved into two components, one perpendicular to  $P$  and one parallel to  $P$ . The one perpendicular to  $P$ ,  $M \cos \theta$ , represents a

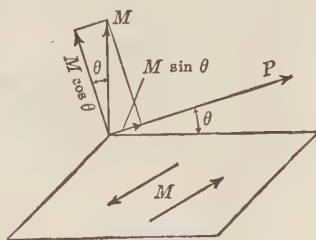


FIG. 116

couple in a plane containing  $P$  and a line in the plane of the couple, perpendicular to the projection of  $P$  upon that plane. This component couple may be replaced by two forces, parallel and equal to  $P$ , and so placed that one of them will annul  $P$ , leaving the other parallel to  $P$ , in the same direction and at a distance  $M \cos \theta / P$  from  $P$ . The other component of the couple,  $M \sin \theta$ , represents a couple in a plane perpendicular to  $P$ . Hence a force and a couple may be reduced to a force parallel and equal to the given force, and a couple in a plane perpendicular to the given force.

**48. Resultant of Parallel Forces.** If the forces of §45 are all parallel, let the  $z$  axis be chosen parallel to the forces. Then for each force,  $X = 0$ ,  $Y = 0$ ,  $Z = P$ , and the resultant force at the origin becomes  $R = \Sigma P$ , parallel to the forces, and the moments

of the resultant couples in the coordinate planes become

$$M_x = \Sigma yP,$$

$$M_y = -\Sigma xP,$$

$$M_z = 0.$$

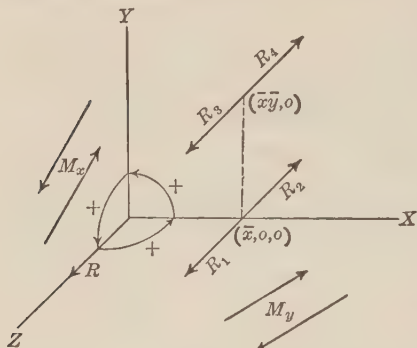


FIG. 117

The resultant force  $R$  at  $O$  and the resultant couples  $M_x$  and  $M_y$  are represented in Fig. 117. The force  $R$  and the couples  $M_x$  and  $M_y$  may

be replaced by a single force as follows: At  $(\bar{x}, 0, 0)$  and  $(\bar{x}, \bar{y}, 0)$ , put in pairs of equal and opposite forces equal to  $R$ , calling them  $R_1, R_2, R_3, R_4$  as shown. Then  $R$  and  $R_2$  form a couple of moment  $\bar{x}R$  in the  $xz$  plane, and  $R_1$  and  $R_4$  form a couple of moment  $-\bar{y}R$  in the  $yz$  plane, leaving the force  $R_3$  acting through  $(\bar{x}, \bar{y}, 0)$  parallel to the  $z$  axis and equal to  $\Sigma P$ . Now let  $\bar{x}$  and  $\bar{y}$  be chosen so that  $\bar{x}R = \Sigma xP$  and  $\bar{y}R = \Sigma yP$ . The couples  $\bar{x}R$  and  $-\bar{y}R$  will then annul the couples  $M_y$  and  $M_x$ , respectively, and the resultant of the original set of forces is the one force  $R = \Sigma P$ , acting through the point  $(\bar{x}, \bar{y}, 0)$ . Hence, if a set of parallel forces act perpendicular to the  $xy$  plane, the resultant of the forces is a force equal to the algebraic sum of the given forces, parallel to them and passing through the point  $(\bar{x}, \bar{y}, 0)$  determined by the equations

$$\bar{x}\Sigma P = \Sigma xP, \quad \bar{y}\Sigma P = \Sigma yP,$$

where  $P$  represents any force of the set, and  $(x, y, 0)$  is a point through which its line of action passes.

In what precedes, the  $x$  axis may be any line perpendicular to the forces, and the distance from the resultant to this axis is determined by the equation

$$\bar{y} = \Sigma yP / \Sigma P.$$



**49. Illustration.** Weights of 60 lbs., 80 lbs., and 100 lbs. act at the vertices of a triangle of sides 5', 7', and 9', perpendicular to the plane of the triangle as shown in Fig. 118; to find the resultant and the position of the line in which it acts.

The perpendiculars from the opposite vertices to the 7' side and the 9' side are found by computation to be 4.97' and 3.87' respectively. The perpendicular distances of the line of the resultant from the lines  $BC$  and  $AC$  are then

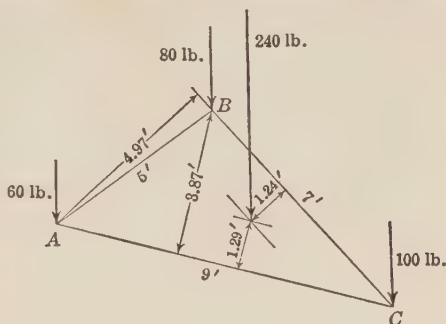


FIG. 118

$$\frac{60 \cdot 4.97 + 80 \cdot 0 + 100 \cdot 0}{60 + 80 + 100} = 1.24' \quad \text{and} \quad \frac{80 \cdot 3.87}{240} = 1.29'$$

respectively. The value of the resultant is  $60 + 80 + 100 = 240$  lbs.

### PROBLEMS

82. Weights of 100 lbs., 150 lbs. and 125 lbs. act at the vertices  $A$ ,  $B$ , and  $C$  respectively of a triangle, perpendicular to the plane of the triangle. The lengths of the sides are  $AB = 3'$ ,  $BC = 5'$ ,  $AC = 6'$ . Find the value of the resultant and the distances of its line of action from  $AB$  and  $BC$ .

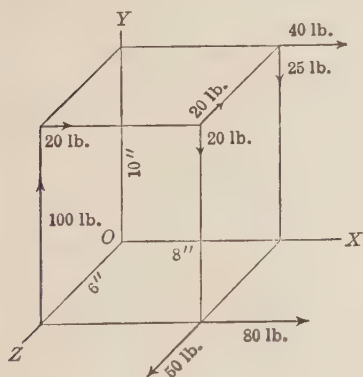


FIG. 119

83. Replace the set of forces shown in Fig. 119 by a single force acting at  $O$  and a single couple, determining the direction angles of the resultant force vector and the direction angles of the normal to the plane containing the resultant couple.

84. A force of 50 lbs. makes an angle of  $30^\circ$  with a plane in which two forces of 20 lbs. each act in opposite directions in lines 6 in. apart. Show that this force and couple may be replaced by a force of 50 lbs., parallel to the original position, but distant from that position 2.08 in. in a direction parallel to the plane and perpendicular to the projection of the force on the plane, and a

couple of 60 lb.-in. in a plane perpendicular to the force.

85. Three vertical ropes support a horizontal triangular slab by being attached to the vertices  $A$ ,  $B$ , and  $C$  of the slab. The edges of the slab are



89. In the shear-legs of Fig. 123, find the tension in the backstay  $AD$  and the compression in the legs  $AB$  and  $AC$ .

(SUGGESTION. Replace the forces in the legs by a single force and thus reduce the forces to a coplanar set.)

90. Find the tension in the backstay and the compressions in the legs of the shear-legs of Fig. 124.

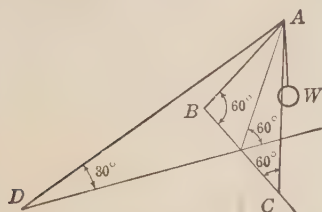


FIG. 123

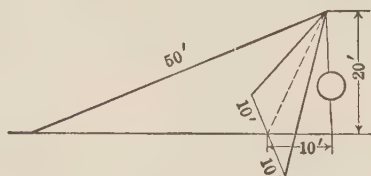


FIG. 124

91. In Fig. 125, the four supporting legs are in vertical planes in pairs. Find the compression in each leg.

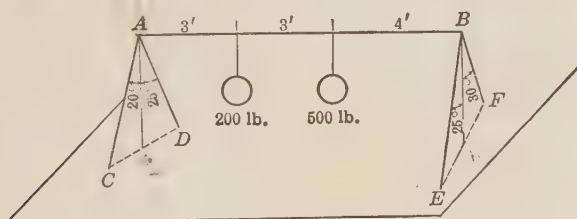


FIG. 125

92. Find the tension  $T$  and the horizontal and vertical components of the reactions of the bearings,  $A$  and  $B$ , for equilibrium in Fig. 126.

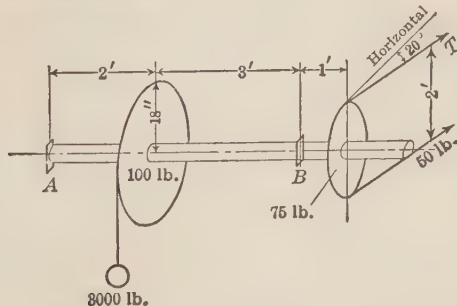


FIG. 126

93. In Fig. 127, a weight  $W$  is supported on the legs of a tripod  $AD$ ,  $BD$ ,  $CD$ , the ends  $A$ ,  $B$ ,  $C$  being in the same horizontal plane. Having given the lengths of the legs and the sides of the triangle  $ABC$ , it is required to find the forces in the legs.



96. A weight of 2000 lbs. is suspended at a point  $D$  from three ropes, the other ends of the ropes being attached to the vertices of a triangle  $ABC$  in a horizontal plane. If  $AB = 10'$ ,  $AC = 6'$ ,  $BC = 8'$ , and the lengths of the ropes are  $AD = 20'$ ,  $DB = 20'$ ,  $DC = 19.6'$ , find the tensions in the ropes. Solve graphically and check one of the forces by an algebraic solution.

97. A table with three legs equally spaced and distant 3 feet apart, weighs 60 lbs. and its weight rests equally on the legs. How far from the center of the table can a weight of 120 lbs. rest without tipping the table?

98. A uniform ladder 13 ft. long, weighing 25 lbs., is placed on top of a box as shown in Fig. 130, the weight of the box being 40 lbs. If the wall is smooth, and the box is prevented from slipping out from the wall by a peg in the floor at the outer edge, how far up the ladder may a man weighing 160 lbs. go before the box tips over?

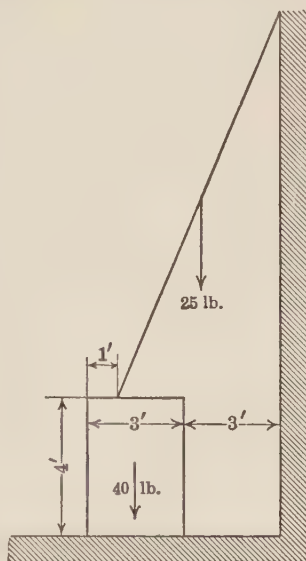


FIG. 130

99. In Fig. 131, a weight of 1000 lbs. is suspended from a T-shaped frame and is supported by three vertical ropes as shown. The stem of the frame makes an angle of  $20^\circ$  with the horizontal and the crossbar an angle of  $15^\circ$  with the horizontal. Find the tensions in the ropes. If the frame were placed in a horizontal position and the ropes were kept vertical, would the tensions remain unchanged?

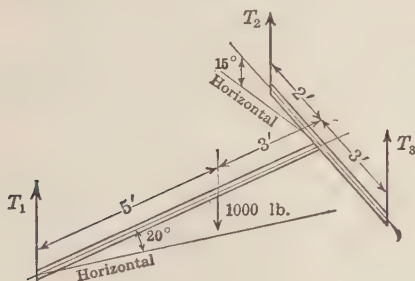


FIG. 131

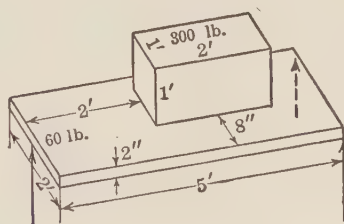


FIG. 132

100. A rectangular plank, 2" thick, 2' by 5', weighing 60 lbs., is placed in a horizontal position and supported at two corners of one end and at the middle point of the lower edge of the opposite end, as shown in Fig. 132. A uniform rectangular block, weighing 300 lbs., with dimensions 2'  $\times$  1'  $\times$  1', is placed with one edge 8" from the edge of the plank and one end 2' from the end of the plank that is supported by the single force. Find the values of the supporting forces. Solve if the right end of the plank is tipped up through  $30^\circ$ .

## CHAPTER III

### PARALLEL FORCES, CENTROIDS

**50. Center of Parallel Forces.** It was shown in §18 that the resultant of two parallel forces is a force equal to their algebraic sum, parallel to the forces, and divides a line joining their lines of action in the inverse ratio of the forces. In Fig. 133,  $P_1$  and  $P_2$  are two parallel forces with points of application at  $A_1(x_1, y_1, z_1)$

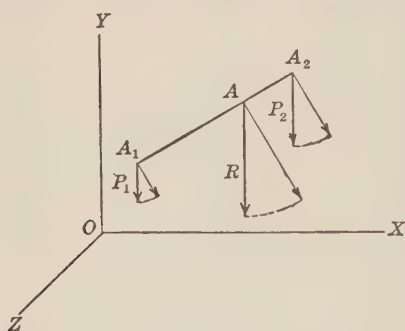


FIG. 133

and  $A_2(x_2, y_2, z_2)$ . It follows that their resultant is  $R = P_1 + P_2$  and its line of action cuts the line  $A_1A_2$  in a point  $A(x, y, z)$  such that  $A_1A/AA_2 = P_2/P_1$ . If  $A_1$  and  $A_2$  are held fast and the forces  $P_1$  and  $P_2$  are kept of the same magnitude and parallel to each other, but are changed in direction, the point  $A$  will not change, since

there is only one point which divides a given line in a given ratio. The resultant would then turn about the point  $A$  as the direction of the forces was changed. The point  $A$  is called the *center* of the two forces.

If lines are drawn from  $A_1$ ,  $A$ , and  $A_2$  parallel to the  $y$  axis to intersect the  $xz$  plane in  $B_1(x_1, 0, z_1)$ ,  $B(x, 0, z)$  and  $B_2(x_2, 0, z_2)$  and these points are projected upon a line  $B_1DD_2$ , parallel to the  $x$  axis, Fig. 134, the following equations hold:

$$\frac{P_2}{P_1} = \frac{A_1A}{AA_2} = \frac{B_1B}{BB_2} = \frac{B_1D}{DD_2} = \frac{x - x_1}{x_2 - x}.$$

Therefore

$$x_2P_2 - xP_2 = xP_1 - x_1P_1$$

or

$$x(P_1 + P_2) = x_1P_1 + x_2P_2.$$



If now a third force,  $P_3$ , act at  $(x_3, y_3, z_3)$  parallel to  $P_1$  and  $P_2$ , it may be combined in the same manner with the resultant of  $P_1$  and  $P_2$  to obtain the resultant of the three forces. The value of this resultant is  $P_1 + P_2 + P_3$ , by § 18. If its line of action intersects the line joining  $A(x, y, z)$  and  $A_3(x_3, y_3, z_3)$  in a point  $C(\bar{x}, \bar{y}, \bar{z})$ , then by the above process,

$$\bar{x}[(P_1 + P_2) + P_3] = x(P_1 + P_2) + x_3P_3.$$

But

$$x(P_1 + P_2) = x_1P_1 + x_2P_2.$$

Therefore

$$\bar{x}(P_1 + P_2 + P_3) = x_1P_1 + x_2P_2 + x_3P_3.$$

By this process the formula can be extended to any number of parallel forces acting at definite points. The formulas for the  $y$  and  $z$  coordinates of the center of forces may be written by analogy. Thus if parallel forces  $P_1, P_2, \dots, P_n$  act at points  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$ , their resultant is  $P_1 + P_2 + \dots + P_n$  and it passes through  $(\bar{x}, \bar{y}, \bar{z})$ , where

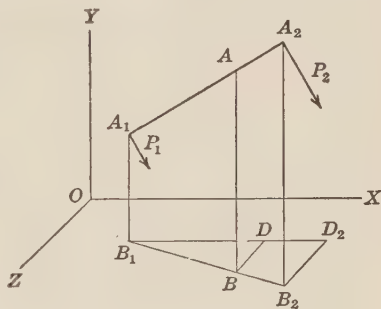


FIG. 134

$$\bar{x}(P_1 + P_2 + \dots + P_n) = x_1P_1 + x_2P_2 + \dots + x_nP_n,$$

$$\bar{y}(P_1 + P_2 + \dots + P_n) = y_1P_1 + y_2P_2 + \dots + y_nP_n,$$

$$\bar{z}(P_1 + P_2 + \dots + P_n) = z_1P_1 + z_2P_2 + \dots + z_nP_n,$$

or, in abbreviated form,

$$\bar{x}\Sigma P = \Sigma xP, \quad \bar{y}\Sigma P = \Sigma yP, \quad \bar{z}\Sigma P = \Sigma zP.$$

It may be noted that the first two of these formulas were obtained in § 48, where the parallel forces were parallel to the axis of  $z$ .

In the above formulas any of the coordinates of the points may be negative without affecting their validity, as is evident from the method of derivation. If the forces are not all in the same direction, those acting in one direction must be counted positive and those in the opposite direction negative.

**51. Moment of the Resultant of a Set of Parallel Forces.** If the forces are all parallel to the  $y$  axis, the equation  $\bar{x}\Sigma P = \Sigma xP$  may be interpreted as follows: The left member is the moment of the resultant of the forces with respect to the  $z$  axis and the right member is the sum of the moments of the separate forces with respect to the same axis. If the forces are not parallel to the  $y$  axis let the direction cosines of the forces be  $l, m, n$ , and let the components of any force  $P$  be  $X, Y, Z$ . By § 44 the moment of  $P$  with respect to the  $z$  axis is  $xY - yX$ . But  $X = Pl$ ,  $Y = Pm$ , and since  $l$  and  $m$  are the same for all of the forces, the sum of the moments of all the forces with respect to the  $z$  axis is

$$m\Sigma xP - l\Sigma yP.$$

Substituting for  $\Sigma xP$  and  $\Sigma yP$  the values  $\bar{x}\Sigma P$  and  $\bar{y}\Sigma P$  respectively from § 50, the equation is obtained

$$m\Sigma xP - l\Sigma yP = \bar{x}m\Sigma P - \bar{y}l\Sigma P.$$

Since  $\Sigma P$  is the resultant of all the forces,  $l\Sigma P$  and  $m\Sigma P$  are respectively the components of the resultant parallel to the  $x$  and  $y$  axis, and the right member therefore represents the moment of the resultant with respect to the  $z$  axis (§ 44). Since the  $z$  axis is any line, the equation states the following theorem:

*The sum of the moments of any set of parallel forces with respect to any line is equal to the moment of their resultant with respect to that line.*

**52. Center of Gravity of a Body. Center of Mass.** If a body be divided up into parts of small volume, the weights of these parts may be regarded as practically parallel forces acting at points in the respective parts. The resultant of these forces is the weight of the body and the center of the forces could be computed by the method of § 50. Thus, if  $w$  is the average weight per unit volume of an elementary volume  $\Delta v$  of the body and  $(x, y, z)$  is a point at which the weight of this element could be regarded as acting, the coordinates of the center of the forces would be given by

$$\bar{x}\Sigma w \Delta v = \Sigma x w \Delta v, \quad \bar{y}\Sigma w \Delta v = \Sigma y w \Delta v, \quad \bar{z}\Sigma w \Delta v = \Sigma z w \Delta v.$$

If now the size of each elementary volume is made to approach zero as a limit, the limiting position of the center of the forces is called the *center of gravity* of the body. We thus get for the coordinates of the center of gravity of the body,

$$\bar{x} = \frac{\int x w dv}{\int w dv}, \quad \bar{y} = \frac{\int y w dv}{\int w dv}, \quad \bar{z} = \frac{\int z w dv}{\int w dv},$$

where  $w$  is the weight per unit volume of the body at the point  $(x, y, z)$ . If  $w$  is constant it may be cancelled out of the equations.

If instead of the weights of the elements their masses are used, the point obtained by the formulas is called the *center of mass* of the body, or its *centroid*. Thus, if  $dm$  is the mass of a small element of the body and  $(x, y, z)$  the point at which it may be thought of as concentrated, the coordinates of the centroid are determined by the equations,

$$\bar{x} \int dm = \int x dm, \quad \bar{y} \int dm = \int y dm, \quad \bar{z} \int dm = \int z dm,$$

or since  $\int dm = M$ , the mass of the whole body, the centroid formulas become

$$\bar{x} = \frac{\int x dm}{M}, \quad \bar{y} = \frac{\int y dm}{M}, \quad \bar{z} = \frac{\int z dm}{M}.$$

Since the weights of the elements are practically proportional to their masses, for bodies outside the earth and near its surface, the centroid of a body practically coincides with the center of gravity. The same symbols are used to represent the coordinates of centroid and center of gravity.

**53. Centroid of an Area.** If the element of area,  $dA$ , is used instead of the element of mass, the point obtained by the formulas is called the *centroid of the area*. A like extension of the formulas is made to include the definition of the centroid of an arc, the element of arc taking the place of the element of mass in the formulas for the center of mass.

It helps to give one a physical concept of the process of finding the centroid of an area, for example, to think of the area as replaced by a very thin lamina of uniform thickness and density and the elements of area would then be replaced by forces

(weights) proportional to these areas and acting at their centers, or approximate centers. These forces may be thought of as acting parallel to any coordinate axis without changing the position of the center of the forces. The moment equation of § 50 may then be applied and the formula for one of the coordinates of the center of forces written. In cancelling out the weight per unit area of surface, introduced in regarding the area as a thin lamina, the formula reduces to that of the centroid of the area.

**54. Illustrations.** **EXAMPLE 1.** As an illustration, suppose it to be required to find the coordinates of the centroid of the area bounded by a parabola  $y^2 = mx$ , the  $x$  axis, and a line  $x = a$ .

**FIRST METHOD.** Divide the given area up into strips parallel to the  $y$  axis, each of width  $dx$ , Fig. 135. Let  $(x, y)$  be a point on the curve through which one of the division lines passes. The strip adjacent to this division line may be thought of as having weight  $w y dx$  and as acting at the point  $(x, y/2)$ . To find the value of  $x$  of the centroid think of these weights as acting parallel to the  $y$  axis and take moments about a line through the origin and perpendicular to the plane of the

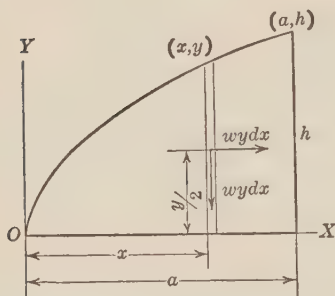


FIG. 135

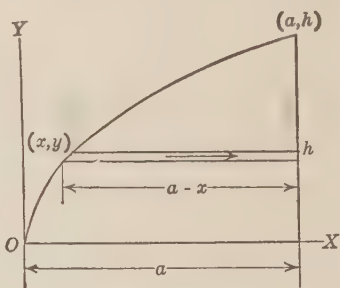


FIG. 136

area. Applying the theorem that the moment of the resultant is equal to the sum of the moments of the separate forces, the equation for the determination of  $x$  is

$$\bar{x} \int_0^a w y dx = \int_0^a x w y dx.$$

Since in this equation  $y$  is a function of  $x$ , namely,  $y = \sqrt{mx}$ , the value of  $y$  in terms of  $x$  must be substituted before the integration is per-

formed. Cancelling  $w$ , and making the substitution, we find

$$\bar{x} \int_0^a \sqrt{mx} \, dx = \int_0^a x \sqrt{mx} \, dx,$$

from which  $\bar{x} = 3a/5$ .

To find  $\bar{y}$ , if we assume the forces to act parallel to the  $x$  axis and take moments as before, we obtain

$$\bar{y} \int_0^a w y \, dx = \int_0^a \frac{y}{2} w y \, dx.$$

Substituting for  $y$ , and noting that  $\sqrt{ma} = h$ , the value of the ordinate of the parabola where  $x = a$ , we may reduce the equation to  $\bar{y} = 3h/8$ . A different equation for the determination of  $\bar{y}$  may be obtained by dividing the area up into strips parallel to the  $x$  axis, Fig. 136. The length of the strip is then  $a - x$  and, if the weights of these strips be regarded as acting horizontally, the moment arm is  $y$ . The equation for the determination of  $\bar{y}$  then becomes

$$\bar{y} \int_0^h (a - x) w \, dy = \int_0^h y(a - x) w \, dy.$$

The result is, of course, the same as that already obtained. Likewise a different equation for the determination of  $\bar{x}$  can be obtained from Fig. 136 by thinking of the weights of the strips as acting vertically, the moment arm then being  $x + \frac{1}{2}(a - x)$  or  $\frac{1}{2}(a + x)$ . The equation for determining  $\bar{x}$  is then

$$\begin{aligned} \bar{x} \int_0^h (a - x) w \, dy \\ = \int_0^h \frac{1}{2} (a^2 - x^2) w \, dy. \end{aligned}$$

**SECOND METHOD.** Divide the area up into elementary rectangles by lines parallel to the axes, Fig. 137. Letting the dimensions of a typical rectangle be  $dx$  and  $dy$ , and the coordinates of a point in this rectangle be  $x$  and  $y$ , we may write the moment equations in the form

$$\begin{aligned} \bar{x} \int_0^a \int_0^{\sqrt{mx}} w \, dx \, dy &= \int_0^a \int_0^{\sqrt{mx}} x w \, dx \, dy, \\ \bar{y} \int_0^a \int_0^{\sqrt{mx}} w \, dx \, dy &= \int_0^a \int_0^{\sqrt{mx}} y w \, dx \, dy, \end{aligned}$$

in which the integration is first performed with respect to  $y$ , holding  $x$

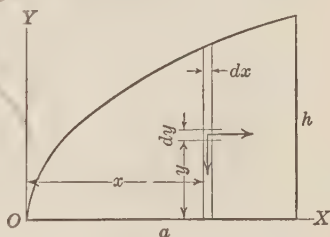


FIG. 137



constant, and the limits 0 and  $\sqrt{mx}$  substituted for  $y$  before integrating with respect to  $x$ . The results of the first integration are

$$\bar{x} \int_0^a \sqrt{mx} \, dx = \int_0^a x \sqrt{mx} \, dx,$$

$$\bar{y} \int_0^a \sqrt{mx} \, dx = \frac{1}{2} \int_0^a mx \, dx,$$

which are the same as the equations for the determination of  $\bar{x}$  and  $\bar{y}$  found by the first method when the vertical strips were used.

Here, again, the strips might have been taken horizontally, in which case the integration would have been performed first with respect to  $x$  between the limits  $y^2/m$  and  $a$ , after which the integration would be carried out for  $y$  between the limits 0 and  $h$ .

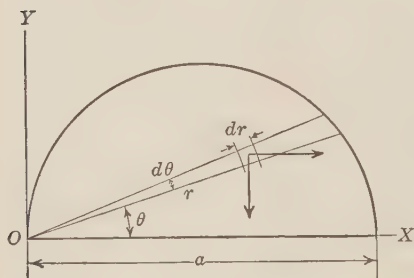


FIG. 138

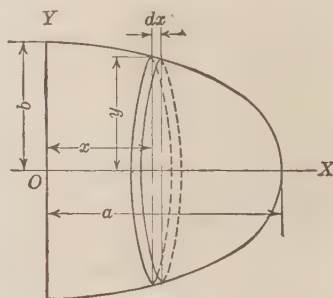


FIG. 139

EXAMPLE 2. As a second illustration let it be required to find the centroid of the area bounded by the curve  $r = a \cos \theta$ , the line  $\theta = 0$ , and the line  $\theta = \pi/2$ , Fig. 138.

Using polar coordinates, we may divide the area up into elements of area  $rd\theta \, dr$ . Let  $(r, \theta)$  be a point in any element of area. Assuming the area to carry a weight  $w$  per unit area and letting it act first parallel to the  $y$  axis and then parallel to the  $x$  axis, we may write the moment equations for the determination of  $\bar{x}$  and  $\bar{y}$  in the form

$$\bar{x} \int_0^{\pi/2} \int_0^{a \cos \theta} wrd\theta \, dr = \int_0^{\pi/2} \int_0^{a \cos \theta} wr \cos \theta r d\theta \, dr,$$

$$\bar{y} \int_0^{\pi/2} \int_0^{a \cos \theta} rd\theta \, dr = \int_0^{\pi/2} \int_0^{a \cos \theta} r \sin \theta r d\theta \, dr.$$

The results are  $\bar{x} = a/2$ ,  $\bar{y} = 2a/(3\pi)$ .

EXAMPLE 3. To locate the centroid of the solid enclosed by the surface generated by revolving the ellipse  $x^2/a^2 + y^2/b^2 = 1$  around the  $x$  axis between  $x = 0$  and  $x = a$ , Fig. 139.



Divide the solid up into disks of thickness  $dx$  by planes perpendicular to the  $x$  axis. Let  $x$  be the distance of any disk from the origin and  $y$  the radius of this disk. Taking moments about a line through  $O$  perpendicular to the  $xy$  plane, we may write the equation for the determination of  $\bar{x}$  in the form

$$\bar{x} \int_0^a y^2 dx = \int_0^a x y^2 dx.$$

Here  $y^2 = b^2(a^2 - x^2)/a^2$ , and on substituting this value and integrating, we find the result  $\bar{x} = 3a/8$ . On account of symmetry,  $\bar{y} = 0$ .

### PROBLEMS

101. Parallel forces of 80 lbs., 65 lbs., 20 lbs. and 40 lbs. act at  $(0, 1, 2)$ ,  $(3, 0, 1)$ ,  $(2, 1, 1)$  and  $(3, -1, 0)$  respectively, in the same direction. Find the magnitude of the resultant and the center of the forces.

102. Reverse the direction of the 65-lb. force in the preceding problem and find the value of the resultant and the center of the forces.

103. Find the centroid of the area of Fig. 136, using horizontal strips.

104. Find the centroid of the area of Fig. 136 by double integration, integrating first with respect to  $x$ .

105. Prove that the centroid of the area of a quadrant of a circle of radius  $r$  is distant  $4r/(3\pi)$  from each of the bounding radii.

106. Prove that the centroid of a uniform hemisphere of radius  $r$  is distant  $3r/8$  from the center of the sphere.

107. A sector of a circle of radius  $r$  has a central angle of  $60^\circ$ . Prove that the distance of its centroid from the center of the circle is  $2r/\pi$ . Use polar coordinates.

108. Prove that the centroid of a right circular cone of altitude  $h$  is distant  $h/4$  from its base.

109. Prove that the centroid of the area of a triangle is distant one-third of the altitude from the base. Show also that it lies at the intersection of the medians of the triangle.

110. Three equal weights are placed at the vertices of a triangle. Prove that their resultant passes through the intersection of the medians of the triangle.

111. Prove that the centroid of the arc of a semi-circle of radius  $r$  is distant  $2r/\pi$  from the center of the circle.

112. Prove that the centroid of a hemispherical surface of radius  $r$  is distant  $r/2$  from the center of the sphere.

113. Prove that the centroid of the surface of a sphere contained between two parallel planes is midway between the planes.

114. A beam 12 ft. long is supported at the ends and carries a load which increases uniformly from 0 at one end to 600 lbs. per linear foot at the other. Find the distance of the resultant load from the middle of the beam. If the beam weighs 25 lbs. per linear foot, what are the reactions of the supports?

115. A beam 20 ft. long, supported at the ends, weighs 30 lbs. per linear foot and carries a load which increases from 200 lbs. per ft. at one end to 600 lbs. per ft. at the other. Find the reactions of the support.

116. A circular sector having a central angle of  $60^\circ$  is revolved about one of the bounding radii to enclose a spherical sector. Find the distance from the center of the circle to the centroid of the spherical sector.

(SUGGESTION. Use polar coordinates and take the ring generated by the element of area  $r d\theta dr$  as the element of weight.)

117. Prove that the centroid of the curved surface of a right circular cone is distant one-third the altitude of the cone from the base.

118. Prove that the centroid of any pyramid is distant one-fourth the altitude from the base.

119. A post of circular cross-section tapers uniformly from a radius of 1 ft. at the base to a radius of 6 in. at the top. The height is 8 ft. How far from the base is the centroid?

120. The area bounded by the parabola  $y^2 = mx$ , the  $x$  axis and the line  $x = a$  is revolved around the  $x$  axis. Find the distance from the vertex of the parabola to the centroid of the solid enclosed.

121. The pressure at any point on a surface submerged in a liquid varies directly as the distance of the point below the surface of the liquid and the pressure upon a unit area at a depth  $h$  below the surface is  $h$  times the weight of a unit volume of the liquid. If a rectangle of base  $b$  and depth  $d$  is submerged in water with the face vertical and upper base in the surface of the water, find the total pressure on one face and the depth of the point at which the resultant force acts. This point is called the center of pressure. (The weight of water is approximately 62.5 lbs. per cu. ft.)

122. A rectangular opening in a vertical dam is 6 ft. wide and 3 ft. deep. If the top of the opening is 10 ft. below the surface of the water, what is the total pressure on the gate which closes the opening and how far from the top of the gate is the center of pressure?

123. Prove that the depth below the surface of the center of pressure on a triangle submerged in water in a vertical plane with vertex in the surface and the base parallel to the surface is three fourths the altitude of the triangle. What is the total pressure on one face of the triangle?

124. Prove that the depth below the surface of the center of pressure on a triangular area submerged in water in a vertical plane and with base in the surface is one-half the altitude of the triangle.

125. Prove that if a rectangular area of altitude  $h$  and base  $b$  is submerged in water so that it is in a vertical plane with base horizontal and center a distance  $d$  below the surface, the center of pressure is at a distance  $h^2/(12d)$  below the center of the rectangle. How does this distance change as the depth of submergence increases? Show that the total pressure on the area is  $wdbh$  where  $w$  is the weight of a unit volume of water.

126. Find the total pressure and the depth of the center of pressure on a circular area of diameter 3 ft. submerged in water in a vertical plane with its center 10 ft. below the surface.

127. Find the total pressure and the depth of the center of pressure on a circular area of radius  $r$  ft. when submerged in water in a vertical plane with its center  $h$  ft. below the surface.

128. Given that two particles of masses  $m_1$  and  $m_2$ , distant  $d$  apart, attract each other with a force equal to  $km_1m_2/d^2$ , prove that if a particle of mass  $m$  is placed in the line of a slender rod of mass  $M$  and length  $l$ , at a distance  $a$  from the nearer end, the total attraction of the rod on the particle is  $kmM/[a(a+l)]$ . (See Fig. 140.) Find also what the attraction would be if the whole mass of the rod were concentrated at its center, and compare the two values.

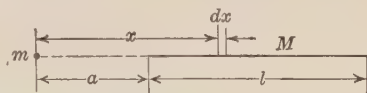


FIG. 140

55. Centroid of a Body with Portion Removed. If a portion of a body is removed, the centroid of the remaining portion may be found by regarding the weight of the remaining part as the

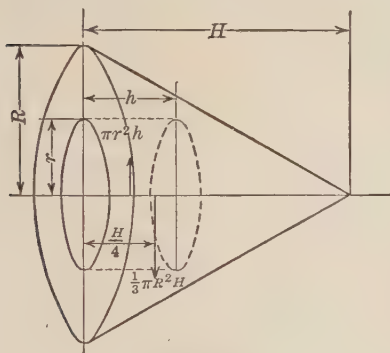


FIG. 141

resultant of the whole body acting at its centroid and the weight of the removed portion acting at its centroid in the reversed direction. The moment of the resultant is then the algebraic sum of the moments of the whole body and the removed portion. For example, if a cylindrical portion be removed from a cone, as indicated in

Fig. 141, in which the axis of the cylinder coincides with the axis of the cone, the distance of the centroid of the remaining portion from the base of the cone is

$$\bar{x} = \frac{\frac{H}{4} \pi \frac{R^2 H}{3} - \frac{h}{2} \pi r^2 h}{\pi \frac{R^2 H}{3} - \pi r^2 h} = \frac{R^2 H^2 - 6r^2 h^2}{4(R^2 H - 3r^2 h)}.$$

(See Problem 108 for the position of the centroid of the cone.)

## PROBLEMS

129. Prove that the removal of the cylindrical portion from the cone, as in Fig. 141, moves the centroid away from the base, toward the base, or leaves it unchanged according as  $h$  is less than  $H/2$ , greater than  $H/2$ , or equal to  $H/2$ .

130. Regarding the "T" section of Fig. 142 as composed of two rectangles, find the distance of the centroid from the top of the "T."

131. Making use of the position of the centroids of a quadrant of a circle and a triangle, found in Problems 105 and 109, find, without integration, the distances of the centroid of the area of Fig. 143 from  $OX$  and  $OY$ .

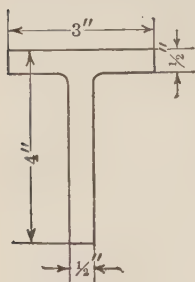


FIG. 142

132. Without integration, find the position of the centroid of the area between a quadrant of a circle and tangents drawn at the end of the quadrant, Fig. 144.

133. Prove that the centroid of the area of a semi-circular ring of radii  $R$  and  $r$  is distant

$$\frac{4}{3\pi} \frac{R^3 - r^3}{R^2 - r^2}$$

from the center of the ring.

134. Prove that the centroid of a hemispherical shell of radii  $R$  and  $r$  is distant

$$\frac{3}{8} \frac{R^4 - r^4}{R^3 - r^3}$$

from the center of the shell.

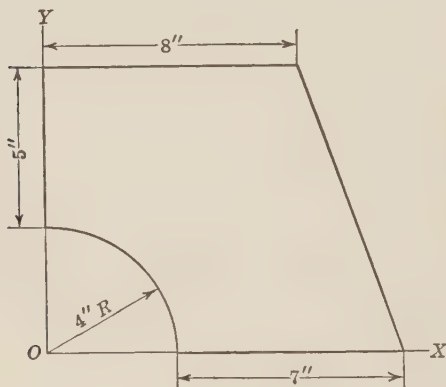


FIG. 143

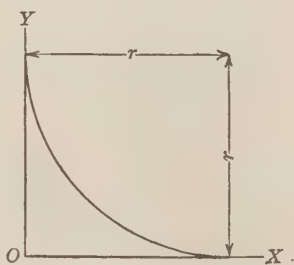


FIG. 144

135. Fig. 145 represents a cubical block of 10-inch edge. In the centers of the two faces indicated holes are bored perpendicular to the faces, 3 inches deep and 2 inches in diameter. Find the coordinates of the centroid of the remaining part.

136. If the holes in the block of the preceding problem are filled with a material of twice the density of the material of the block, find the centroid.

137. A hemispherical hollow cap of radii 10 inches and 6 inches rests upon the top of a hollow cylinder of the same radii and of height 12 inches. Find the height of the centroid above the base of the cylinder.

138. In a circle of radius 10 inches, a chord is drawn at a distance of 6 inches from the center. Find the distance from the center of the circle to the centroid of the area of the segment cut off by the chord. Then combining this segment with the triangle formed by the chord and the radii drawn to its ends, find the distance from the center of the circle to the centroid of the sector bounded by the radii to the ends of the chord and the arc subtending the chord.

139. Solve the preceding problem for a circle of radius  $r$  and a chord drawn at a distance  $d$  from the center.

140. The top of a right circular cone, standing with axis vertical, is removed, leaving a frustum of half the height of the cone. By how much is the centroid lowered?

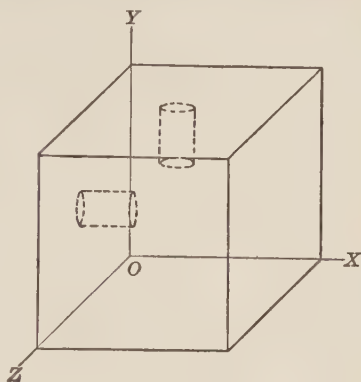


FIG. 145

### 56. Centroid of a Plane Area by Graphical Construction.

When the equations of the boundary curves of a plane area are not known, or when the integration involved in the evaluation of the centroid formulas cannot be carried out, it is sometimes

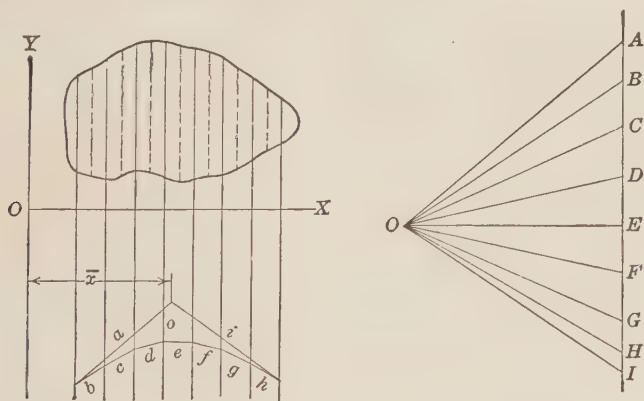


FIG. 146

necessary to use some approximate method of locating the centroid. One such method is by a graphical construction as follows:

Divide the given area up into narrow strips of equal width parallel to the  $y$  axis (Fig. 146). Let  $h_1, h_2, h_3, \dots$  be the



lengths of these strips midway between their sides. Since the widths of the strips are equal the areas are approximately proportional to the  $h$ 's. The location of the centroid of the area then reduces to the finding of the center of a set of parallel forces acting in the lines of the  $h$ 's and proportional to them. This is a special case of finding the line of the resultant of a set of coplanar forces as explained in § 29, the only difference being that here the forces are all parallel. Thus the force polygon  $AB \cdots I$  is laid off in which to a certain scale,  $AB = h_1$ ,  $BC = h_2$ , etc. A point  $O$  is then chosen and  $OA$ ,  $OB$ ,  $\cdots OI$  drawn. Then on the lines of  $h_1$ ,  $h_2$ ,  $\cdots$  the equilibrium polygon is constructed,  $o/a$  parallel to  $OA$ ,  $o/b$  parallel to  $OB$ , etc. The intersection of  $o/a$  and  $o/i$  is then a point on the resultant of the forces acting in the lines of  $h_1$ ,  $h_2$ ,  $\cdots$ ; that is, it is a point in a line parallel to the  $y$  axis through the centroid of the area. The  $\bar{x}$  of the area is thus determined. A similar construction in which the area is divided into strips parallel to the  $x$  axis, will determine the value of  $\bar{y}$  and so locate the centroid of the area.

### PROBLEMS

141. Find by the above construction the location of the centroid of a quadrant of a circle of radius 2 in. Scale the result and compare with the computed value.

142. Find by graphical construction the abscissa of the centroid of the area bounded by the curve  $y^2 = mx$ , the  $x$  axis, and the line  $x = a$ . Scale and compare with the computed value.

143. An oval-shaped area lies between and touches two parallel lines 8 inches apart. The area is divided into 8 strips of equal width by lines parallel to the two lines. The lengths of the middle lines of these strips are in inches, beginning at the left, 2, 3.5, 5, 7.2, 8.6, 8.4, 8, 4. By graphical construction find the distance of the centroid from the left boundary line. Also compute the distance approximately and compare the results.

57. **Simpson's Rule.** It is shown in calculus that the area between a curve  $y = f(x)$ , the  $x$  axis, and the lines  $x = a$  and  $x = b$  is given by the formula

$$A = \int_a^b f(x) dx.$$

If it is desired to find the area under a curve whose equation is not



known, or if the integration of  $f(x)dx$  cannot be carried out, some approximate method may be used. One such widely used method, and in general a fairly accurate one, is known as *Simpson's Rule*.

Let the curve, Fig. 147, represent  $y = f(x)$  between  $x = a$  and  $x = b$ . Divide the area between the curve and the  $x$  axis into an even number,  $n$ , of strips parallel to the  $y$  axis, of equal width  $d$ , and let the ordinates at the edges of these strips be  $y_0, y_1, y_2, \dots, y_n$ . Through the upper ends of the first three ordinates a parabola with axis parallel to the  $y$  axis may be passed, and if the foot,  $O'$ , of the second ordinate is taken temporarily as origin, the equation of the parabola is of the form

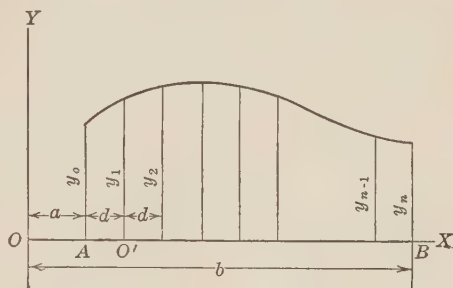


FIG. 147

$$y = px^2 + qx + r,$$

where  $p, q$ , and  $r$  are constants determined by the conditions that the curve passes through the three points  $(-d, y_0), (0, y_1), (d, y_2)$ .

The area of the two strips bounded by the parabola, the  $x$  axis, and the ordinates  $y_0$  and  $y_2$  is then

$$A_{1,2} = \int_{-d}^d (px^2 + qx + r)dx = \frac{2pd^3}{3} + 2rd.$$

Substituting the coordinates of the three points  $(-d, y_0), (0, y_1), (d, y_2)$  in the equation  $y = px^2 + qx + r$ , we have for the determination of  $p$  and  $r$ ,

$$y_0 = pd^2 - qd + r, \quad y_1 = r, \quad y_2 = pd^2 + qd + r.$$

Solving these equations for  $r$  and  $p$ , we find

$$r = y_1, \quad p = (y_0 + y_2 - 2y_1)/2d^2.$$

Substituting these values in the expression for  $A_{1,2}$  we obtain

$$A_{1,2} = \frac{d}{3}(y_0 + 4y_1 + y_2).$$

The area of the two strips under the parabola is, in general, a good approximation to the area of the two strips under the curve, so that the above expression may be regarded as an approximate value of the first two strips under the curve. In the same way, an approximate value of the area of each pair of strips is obtained in terms of  $d$  and the ordinates of the sides of the strips. If all such expressions are added together, there results for the area under the curve the approximate formula,

$$A = \frac{d}{3} [y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n].$$

Since  $A = \int_a^b f(x)dx$ , we have as an approximate value of the integral

$$\int_a^b f(x)dx = \frac{d}{3} [y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n],$$

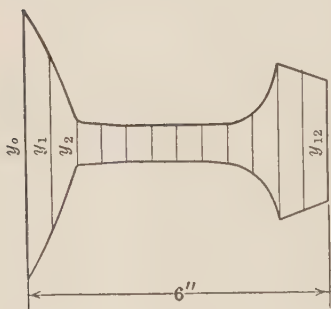


FIG. 148

where  $d = (b - a)/n$ ,  $y_0 = f(a)$ ,  $y_1 = f(a + d)$ ,  $\cdots y_n = f(b)$ ,  $n$  even. In general the larger  $n$  is, the closer the approximation to the true value of the integral.

Since any function of  $x$  may be represented by a curve, the above expression may be used in the evaluation of any integral. For example, in computing centroids of an area by the formula

$$\bar{x} \int y \, dx = \int xy \, dx,$$

the quantity  $xy$  in the second integral takes the place of  $y$ , or  $f(x)$  in the general expression.

In applying the formula it is not necessary that the values of  $y$  be measured from a straight line, since the areas of the strips are determined by their lengths and breadths only.

**58. Illustration.** The area shown in Fig. 148 has the values  $y_0$  to  $y_{12}$  in inches as follows:

$$\begin{array}{lll} y_0 = 5.5, & y_1 = 3.6, & y_2 = 1.0, \\ y_3 = 0.94, & y_4 = y_5 = y_6 = y_7 = 0.88, & \\ y_8 = 0.92, & y_9 = 1.2, & y_{10} = 3.2, \\ y_{11} = 2.8, & y_{12} = 2.5. & \end{array}$$

The area of the section is then given approximately by

$$A = \frac{0.5}{3} [5.5 + 4(3.6 + 0.94 + 0.88 + 0.88 + 1.2 + 2.8) + 2(1.0 + 0.88 + 0.88 + 0.92 + 3.2) + 2.5] = 10.49 \text{ sq. in.}$$

The value of  $\bar{x}$  is given by

$$A\bar{x} = \int_0^6 xy \, dx = \frac{0.5}{3} [0 + 4\{(3.6)(0.5) + (0.94)(1.5) + (0.88)(2.5) + (0.88)(3.5) + (1.2)(4.5) + (2.8)(5.5) + 2\{(1.0)(1) + (0.88)2 + (0.88)(3) + (0.92)(4) + (3.2)5 + (2.5)(6)\}] = 30.39.$$

Therefore

$$\bar{x} = \frac{30.39}{10.49} = 2.90''.$$

### PROBLEMS

**144.** Find the area and  $\bar{x}$  in Fig. 149, given that the vertical distances at equal intervals, beginning at the left, are, in inches, 5, 10.6, 10.8, 10.8, 10.6, 8.4, 6.3, 5.2, 4.6, 3.9, 1.5.

**145.** By Simpson's Rule, taking 10 strips, and computing the ordinates from  $y = (a^2 - x^2)^{1/2}$ , using a 5-place table of square roots, find the area of a quadrant of a circle and the distance of the centroid from one side. (Compare with the correct values.)

**146.** A circular cistern is 10 ft. deep. At intervals of 1 ft., beginning at the top, the radii, in ft., of the cross-sections are respectively 4.0, 4.6, 5.0, 5.4, 5.7, 5.9, 6.0, 6.0, 5.8, 5.5, 5.0. Find the capacity in cu. ft. by Simpson's Rule.

What is the height above the bottom of the centroid of the contents when the cistern is full? (Note that an approximate value of  $\pi \int_0^{10} r^2 dy$  is to be found by Simpson's Rule.)

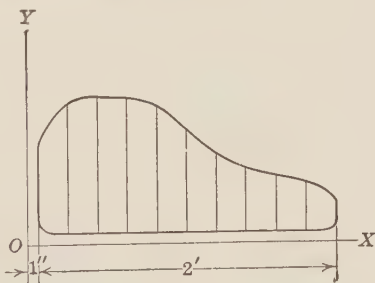


FIG. 149

## CHAPTER IV

### MOMENT OF INERTIA

**59. Moment of Inertia of a Body.** If a body is divided into infinitesimal parts and the mass of each part is multiplied by the square of its distance from a given line, the limiting value of the sum of all such products, as the mass of each infinitesimal part approaches zero, is defined to be the *moment of inertia* of the body with respect to that line.

The symbol  $I$  is usually taken to represent moment of inertia.

**60. Illustrations.** **EXAMPLE 1.** To find the moment of inertia of a circular cylindrical disk of thickness  $t$  and radius  $a$  with respect to a line through the center perpendicular to the face of the disk.

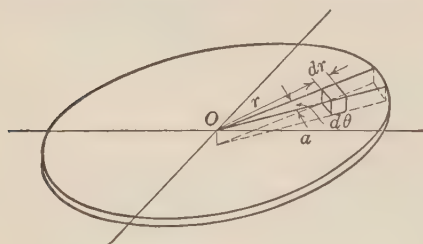


FIG. 150

Using polar coordinates, divide the face of the disk up into elements of area  $r d\theta dr$  and on these elementary areas as bases divide the disk up into elementary volumes  $t r d\theta dr$ , Fig. 150. If  $m$  is the mass per unit volume,

the mass of an element is then  $m t r d\theta dr$ . Multiplying this mass by the square of its distance from the chosen line, and taking the limit of the sum of all the products, we find the value of the moment of inertia of this disk with respect to the chosen line in the form

$$I = \int_0^{2\pi} \int_0^a m t r^3 d\theta dr = \pi m t a^4 / 2.$$

If the mass of the whole disk is  $M$ ,  $M = \pi a^2 m t$ , and the expression for  $I$  may be written

$$I = M a^2 / 2.$$

**EXAMPLE 2.** To find the moment of inertia of a sphere with respect to a diameter.

Let the chosen diameter be taken as the  $x$  axis. There are various ways of choosing the element of mass. A convenient element is that shown in Fig. 151, bounded by planes perpendicular to the  $x$  axis,

distant  $x$  and  $x + dx$  from the center, concentric cylinders around the  $x$  axis of radii  $r$  and  $r + dr$ , and planes containing the  $x$  axis making angles  $\theta$  and  $\theta + d\theta$  with the  $xy$  plane. If  $m$  is the mass of a unit volume, the mass of the element is then  $m r d\theta dr dx$  and its distance from the  $x$  axis is  $r$ , and hence the moment of inertia of the sphere with respect to the  $x$  axis is

$$I_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \int_0^{2\pi} m r^3 dx dr d\theta = \boxed{8\pi m a^5/15},$$

where  $a$  is the radius of the sphere. If the mass of the sphere is  $M$ ,  $M = 4\pi a^3 m/3$ , and the value of  $I_x$  may be written

$$I_x = 2Ma^2/5.$$

A shorter method is to make use of the expression for the moment of inertia of a disk found in the first illustration, namely,  $\pi m t a^4/2$ . For this purpose the sphere may be regarded as cut into slices by planes perpendicular to the  $x$  axis distant  $dx$

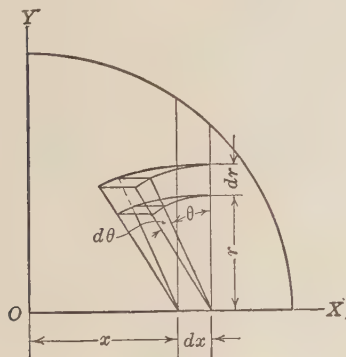


FIG. 151

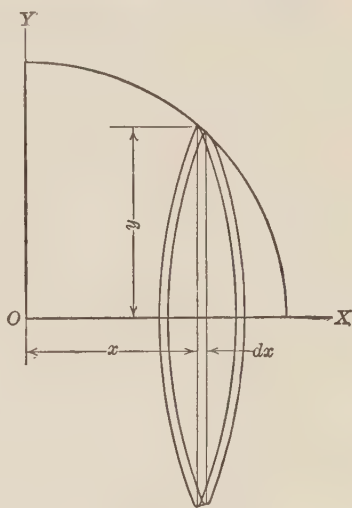


FIG. 152

apart, Fig. 152. For small values of  $dx$ , these slices may be regarded as cylindrical disks with circular faces, variable radii  $y$ , and thickness  $dx$ . The moment of inertia of such a disk with respect to the  $x$  axis is, by the preceding illustration,  $\pi m dxy^4/2$ . Hence the moment of inertia of the whole sphere with respect to the  $x$  axis is

$$I_x = \frac{1}{2} \int_{-a}^a \pi m y^4 dx.$$

But  $y^2 = a^2 - x^2$ , and the substitution of this value reduces  $I_x$  to the form

$$I_x = \frac{\pi m}{2} \int_{-a}^a (a^2 - x^2)^2 dx = \frac{8}{15} \pi m a^5 = \frac{2}{5} M a^2.$$

## PROBLEMS

147. Making use of the formula for the moment of inertia of a circular disk with respect to a line through its center perpendicular to its face, prove that the moment of inertia of a right circular cone with respect to its axis is  $3Mr^2/10$ ,

where  $M$  is the mass and  $r$  is the radius of the base of the cone.

148. Prove that the moment of inertia of a right circular cylinder, of mass  $M$  and radius  $r$ , with respect to the axis of the cylinder is  $Mr^2/2$ .

149. Prove that the moment of inertia of a slab whose faces are rectangles of sides  $a$  and  $b$ , and of thickness  $t$ , with respect to a line through the center perpendicular to the faces, Fig. 153, is

$mt(a^3b + ab^3)/12$ , where  $m$  is the mass per unit volume, or  $M(a^2 + b^2)/12$ , where  $M$  is the mass of the slab.

(SUGGESTION. The element of mass may be taken as  $mt dx dy$ , and its distance from the chosen axis as  $(x^2 + y^2)^{1/2}$ , so that

$$I = \int_{-(b/2)}^{b/2} \int_{-(a/2)}^{a/2} (x^2 + y^2) m t dx dy.$$

150. Using the result of the preceding problem, show that the moment of inertia of a rectangular solid of mass  $M$  and edges  $a$ ,  $b$ , and  $c$ , with respect to an axis through the center parallel to the edge  $c$  is  $M(a^2 + b^2)/12$ .

151. A right pyramid has a rectangular base of sides  $a$  and  $b$  and an altitude  $h$ . Making use of the formula for the moment of inertia of a slab found in Problem 149, show that the moment of inertia of the pyramid with respect to a line through the vertex perpendicular to the base is  $M(a^2 + b^2)/20$ .

152. A solid is enclosed by the surface generated by revolving the parabola  $ay^2 = h^2x$ , between the values  $x = 0$  and  $x = a$ , and the line  $x = a$ , about the  $x$  axis. If the mass of this solid is  $M$ , show that the moment of inertia of this solid with respect to the  $x$  axis is  $Mh^2/3$ .

153. Show that the moment of inertia of a sphere may be obtained by using the element of mass indicated in Fig. 154 and that the value of the moment of inertia of the sphere with respect to a diameter is then

$$I_x = 8 \int_0^a \int_0^{y_1} \int_0^{z_1} m(y^2 + z^2) dx dy dz$$

where  $z_1 = (a^2 - x^2 - y^2)^{1/2}$  and  $y_1 = (a^2 - x^2)^{1/2}$ . Find the value of the integral.

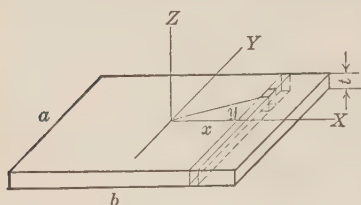


FIG. 153

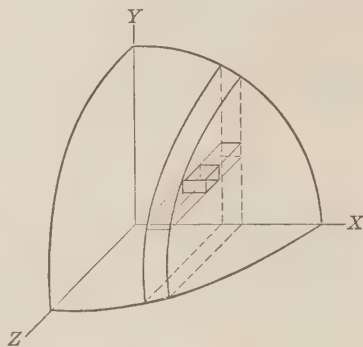


FIG. 154



154. Find the moment of inertia of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with respect to the  $x$  axis.

155. Prove that the moment of inertia of a hollow sphere of uniform density, of mass  $M$ , and with inner and outer radii  $r$  and  $R$ , with respect to a diameter is

$$\frac{2M}{5} \frac{R^5 - r^5}{R^3 - r^3}.$$

### 61. Moment of Inertia of an Area, of an Arc. Definitions.

If each infinitesimal area of a plane area is multiplied by the square of its distance from a given line in the plane of the area, the limit of the sum of all such products, as each element of area approaches zero as a limit, is called the *moment of inertia of the area with respect to the line*.

If each infinitesimal area of a plane area is multiplied by the square of its distance from a given point in the area, the limit of the sum of all such products, as each element of area approaches zero as a limit, is called the *polar moment of inertia of the area with respect to the point*.

If the arc of a plane curve is divided into infinitesimal parts and each part is multiplied by the square of its distance from a given line in the plane of the curve, the limit of the sum of all such products as each element of arc approaches zero as a limit, is called the *moment of inertia of the arc* with respect to the line.

It follows from these definitions that the expressions for the moments of inertia of any plane area with respect to coordinate axes  $OX$  and  $OY$  in the plane become respectively

$$I_x = \int y^2 dA, \quad \text{and} \quad I_y = \int x^2 dA,$$

where  $dA$  is any element of area distant respectively  $y$  and  $x$  from the  $x$  and  $y$  axes. Likewise the polar moment of inertia of the area with respect to the origin is

$$I_0 = \int r^2 dA$$

where  $r$  is the distance of the element of area  $dA$  from the origin. Since  $r^2 = x^2 + y^2$ , it follows at once that

$$I_0 = I_x + I_y.$$

Hence the sum of the moments of inertia of an area with respect to two rectangular axes in the plane of the area is equal to the polar moment of inertia of the area with respect to the intersection of the axes.

## PROBLEMS

156. Prove that the moment of inertia of the area of a rectangle of base  $b$  and altitude  $d$  with respect to the base is  $bd^3/3$ . Solve (a) by single integration, taking the element of area  $b dy$ , Fig. 155 (a); (b) by double integration, taking  $dx dy$  as the element of area, Fig. 155 (b).

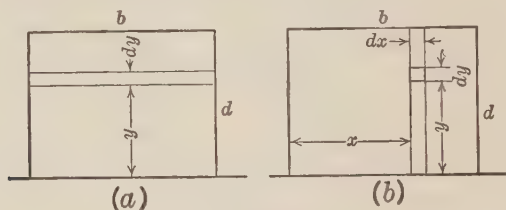


FIG. 155

157. Prove that the polar moment of inertia of the area of a circle with respect to the center is  $\pi a^4/2$ . (Use polar coordinates.)

158. Prove that the moment of inertia of the area of a circle with respect to a diameter is  $\pi a^4/4$ . Obtain the result by integration and also from the result of Problem 157 and the relation  $I_o = I_x + I_y$ .

159. Prove that the moment of inertia of the area of a triangle of base  $b$  and altitude  $h$  with respect to a line through the vertex parallel to the base is  $bh^3/4$ .

160. Prove that the moment of inertia of the area of a triangle of base  $b$  and altitude  $h$  with respect to the base is  $bh^3/12$ .

161. Prove that the moment of inertia of the area of an ellipse with semi-axes  $a$  and  $b$  with respect to the diameter  $2a$  is  $\pi ab^3/4$ .

162. Prove that the moment of inertia of the area of a rectangle of base  $b$  and altitude  $d$  with respect to a line through the center parallel to the base is  $bd^3/12$ .

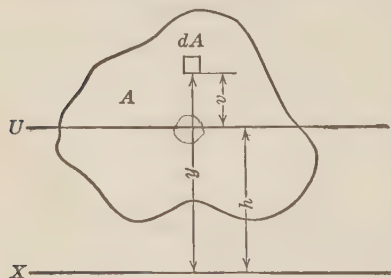


FIG. 156

**62. Relation between Moments of Inertia of an Area with Respect to Parallel Axes.** Let  $X$  and  $U$  be two parallel lines distant  $h$  apart, in the plane of the area, Fig. 156. Let  $I_x$  and  $I_u$  be the moments of inertia of the area  $A$  with

respect to  $X$  and  $U$  respectively. If  $y$  and  $v$  are the ordinates of any element of area  $dA$  with respect to  $X$  and  $U$  respectively, then

$$y = v + h, \quad I_u = \int v^2 dA,$$

and

$$\begin{aligned} I_x &= \int (v + h)^2 dA = \int v^2 dA + 2h \int v dA + h^2 \int dA \\ &= I_u + 2h\bar{v}A + h^2A, \end{aligned}$$

where  $\bar{v}$  is the ordinate of the centroid of  $A$  referred to  $U$ .

If we suppose that  $U$  passes through the centroid,  $\bar{v} = 0$ ; and if we denote  $I_u$  by  $I_c$ , we may write the equation in the form

$$I_x = I_c + h^2A.$$

This formula is useful in transferring the moment of inertia from a centroidal axis to a parallel axis. The same formula may be written

$$I_c = I_x - h^2A,$$

in which form it is more convenient for transferring the moment of inertia from a line to a parallel line through the centroid.

**63. Illustration.** To find the moment of inertia of the area of the "T" section shown in Fig. 157 with respect to the horizontal axis through the centroid.

By moments, the distance of the centroid above the base is

$$\bar{y} = \frac{5(2.5) + 4(5.5)}{5 + 4} = \frac{23}{9}.$$

By Problem 156, the moment of inertia of the area of the whole section (with respect to  $ab$ ) is

$$(4)(1^3)/3 + (1)(5^3)/3 = 43.$$

Then

$$I_c = 43 - 9(7/6)^2 = 30.75.$$

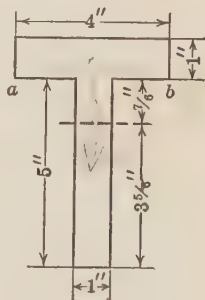


FIG. 157

Since the elements of area are in square inches and the distances are in inches, each term in the sum of the moment of inertia is of the fourth degree in inches. This is indicated by writing

$$I_c = 30.75 \text{ in.}^4$$

If now it is desired to know the moment of inertia of the area of the "T" section with respect to the base line, the result is

$$I_x = I_c + 9(23/6)^2 = 163 \text{ in.}^4$$

### PROBLEMS

163. Having given that the moment of inertia of the area of a triangle of base  $b$  and altitude  $h$  with respect to a line through the vertex parallel to the base is  $bh^3/4$ , prove that the moment of inertia of the area with respect to a centroidal axis parallel to the base is  $bh^3/36$ , and then show that the moment of inertia with respect to the base is  $bh^3/12$ .

164. Prove that the moment of inertia of the area of a circle with respect to a tangent is  $5\pi a^4/4$ , where  $a$  is the radius of the circle. Use the result of Problem 158.

165. Prove that the moment of inertia of the area of a quadrant of a circle with respect to an axis through the centroid parallel to one of the bounding radii is  $0.0549a^4$ , where  $a$  is the radius of the circle.

166. Figure 158 is a sketch of a section of a ship-building channel. Neglecting the rounding of the corners, find the moments of inertia of the area with respect to the centroidal axes parallel and perpendicular to the web.

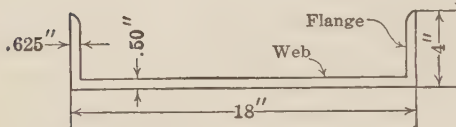


FIG. 158

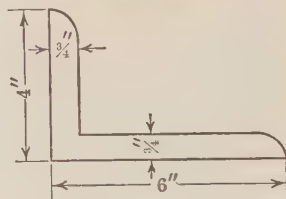


FIG. 159

167. Fig. 159 is a sketch of the section of an angle iron. Neglecting the rounding of the corners, find the moments of inertia with respect to the centroidal axes parallel to the legs.

**64. Radius of Gyration.** *Definition.* The radius of gyration of a body with respect to any line is the distance from the line at which a particle of mass equal to that of the body might be placed and have the same moment of inertia as the body with respect to that line. Thus, if  $I$  is the moment of inertia of a body of mass  $M$  with respect to a given line and  $k$  is the radius of gyration of the body with respect to that line, then, by definition,

$$Mk^2 = I, \quad \text{or} \quad k = \sqrt{I/M}.$$

In the same way, the radius of gyration of an area  $A$  with respect to a line is defined by the equation

$$Ak^2 = I, \quad \text{or} \quad k = \sqrt{I/A}.$$

## PROBLEMS

168. Prove that the radius of gyration of the area of a circle with respect to a diameter is one-half the radius of the circle.

169. Prove that the radius of gyration of a sphere of radius  $a$  with respect to a diameter is  $\sqrt{10}a/5$ .

170. Prove that the radius of gyration of the area of a rectangle of depth  $d$  with respect to the base is  $\sqrt{3}d/3$ .

171. Find the radii of gyration of the area of the angle section of Problem 167 with respect to the centroidal axes parallel to the legs.

**65. Moment of Inertia of an Area with Respect to an Inclined Axis.** In Fig. 160 are shown two sets of rectangular axes having a common origin, the axes  $OX'$  and  $OY'$  being inclined at an angle  $\alpha$  with  $OX$  and  $OY$  respectively. It is useful to have formulas which express the moments of inertia of an area in the plane of the axes with respect to  $OX'$  and  $OY'$  in terms of the moments of inertia with respect to  $OX$  and  $OY$ .

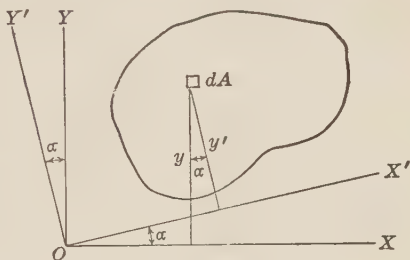


FIG. 160

From the figure it is seen that if  $(x, y)$  and  $(x', y')$  are the coordinates of any point of the plane referred to the axes  $OX, OY$ , and  $OX', OY'$ , respectively, then

$$x' = x \cos \alpha + y \sin \alpha, \quad y' = y \cos \alpha - x \sin \alpha.$$

Then, if  $I_x, I_y, I_{x'}, I_{y'}$  are the moments of inertia of the given area with respect to  $OX, OY, OX', OY'$  respectively,

$$\begin{aligned} I_{x'} &= \int y'^2 dA = \int (y \cos \alpha - x \sin \alpha)^2 dA \\ &= \cos^2 \alpha \int y^2 dA - 2 \cos \alpha \sin \alpha \int xy dA + \sin^2 \alpha \int x^2 dA. \end{aligned}$$

Now  $\int y^2 dA = I_x$ ,  $\int x^2 dA = I_y$ , and hence  $I_{x'}$  may be written

$$I_{x'} = \cos^2 \alpha \cdot I_x - \sin 2\alpha \int xy dA + \sin^2 \alpha \cdot I_y.$$

In the same way,

$$I_{y'} = \sin^2 \alpha \cdot I_x + \sin 2\alpha \int xy dA + \cos^2 \alpha \cdot I_y.$$

These are the required formulas of transformation.



It follows from these equations, by addition, that

$$I_x' + I_y' = I_x + I_y,$$

a result that is also evident from the fact that either  $I_x + I_y$  or  $I_x' + I_y'$  is equal to the polar moment of inertia of the area with respect to the origin.

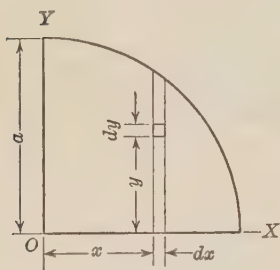


FIG. 161

ment  $dx dy$ , Fig. 161. The product of inertia of the area of the quadrant then becomes

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dx \, dy = \frac{1}{2} \int_0^a x(a^2 - x^2) \, dx = \frac{a^4}{8}.$$

### PROBLEMS

172. Find the product of inertia of the area of a rectangle with respect to two adjacent sides.

173. Find, with respect to  $OX$  and  $OY$ , (a) the product of inertia of the area of the triangle  $OAC$  of Fig. 162; (b) the product of inertia of the area of the triangle  $OCB$  of Fig. 162. Compare their sum with the product of inertia of the rectangle  $OACB$  with respect to  $OX$  and  $OY$ . (c) Find the product of inertia of the area of the triangle  $OAB$ .

174. Using the formula of §65, find the moment of inertia of a rectangle with respect to a diagonal.

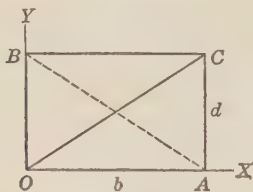


FIG. 162

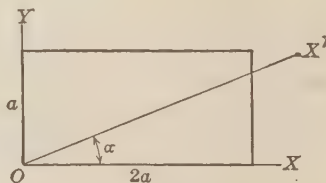


FIG. 163

175. Find the moment of inertia of the area of a quadrant of a circle with respect to a radius bisecting the quadrant and with respect to a line through the center of the circle perpendicular to this radius.



176. Find the moment of inertia,  $I_{x'}$ , of the rectangle of sides  $2a$  and  $a$  with respect to a line through their intersection,  $O$ , making an angle  $\alpha$  with the side  $2a$ . (Fig. 163.) Then regarding  $I_{x'}$  as a function of  $\alpha$ , find the value of  $\alpha$  that makes  $I_{x'}$  a minimum. Show also that a line through  $O$  perpendicular to  $OX'$  thus determined is the line with respect to which the moment of inertia of the area is a maximum for all lines passing through  $O$ .

**67. Maximum and Minimum Moments of Inertia.** In Fig. 164,  $OX$  and  $OY$  are a pair of rectangular axes passing through a point  $O$  in the plane of an area  $A$ , and  $OX'$  is an axis making an angle  $\alpha$  with  $OX$ . If we let  $H = \int xy dA$ , we have from § 65,

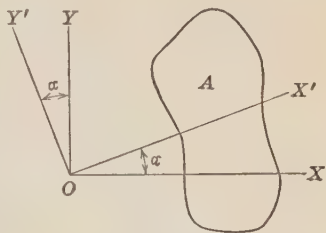


FIG. 164

$$I_{x'} = I_x \cos^2 \alpha - H \sin 2\alpha + I_y \sin^2 \alpha.$$

If  $OX'$  is turned about  $O$ ,  $I_{x'}$  becomes a function of the variable,  $\alpha$ . Taking  $OY'$  perpendicular to  $OX'$ , we have, as shown in § 65,

$$I_{y'} = I_x \sin^2 \alpha + H \sin 2\alpha + I_y \cos^2 \alpha,$$

and

$$I_{x'} + I_{y'} = I_x + I_y.$$

When  $\alpha$  passes continuously through all values from 0 to  $2\pi$ , both  $I_{x'}$  and  $I_{y'}$  pass through all possible values of moments of inertia of the area  $A$  with respect to lines through  $O$ . Both  $I_{x'}$  and  $I_{y'}$  will, in general, change as  $\alpha$  changes and, since their sum remains constant, when one of them takes the maximum value of the set, the other must take the minimum value. Hence the maximum and minimum moments of inertia of the area with respect to lines through  $O$  occur with respect to lines at right angles.

The lines through a point in the plane of an area with respect to which the moments of inertia of the area are maximum and minimum are called the *principal axes* of the area for that point.

To find the location of the principal axes, put the derivative of  $I_{x'}$  with respect to  $\alpha$  equal to zero.

$$\begin{aligned} I_{x'} &= I_x \cos^2 \alpha - H \sin 2\alpha + I_y \sin^2 \alpha, \\ dI_{x'}/d\alpha &= -2I_x \cos \alpha \sin \alpha - 2H \cos 2\alpha + 2I_y \sin \alpha \cos \alpha \\ &= \sin 2\alpha(I_y - I_x) - 2H \cos 2\alpha. \end{aligned}$$

We have also

$$d^2I_x'/d\alpha^2 = 2(I_y - I_x) \cos 2\alpha + 4H \sin 2\alpha.$$

In order for  $I_x'$  to be either a maximum or a minimum,

$$dI_x'/d\alpha = 0;$$

that is, if  $I_x$  is not equal to  $I_y$ ,

$$\tan 2\alpha = \frac{2H}{I_y - I_x}.$$

There are two values of  $2\alpha$ , differing by  $180^\circ$ , that satisfy this equation. If one of these values of  $2\alpha$  is selected,  $\sin 2\alpha$  and  $\cos 2\alpha$  have definite signs and  $d^2I_x'/d\alpha^2$  takes a definite value and sign. If the other value of  $2\alpha$  is selected, the values of  $\sin 2\alpha$  and  $\cos 2\alpha$  will be reversed in sign and therefore the sign of  $d^2I_x'/d\alpha^2$  will be changed. If  $d^2I_x'/d\alpha^2$  is positive when  $dI_x'/d\alpha = 0$ , the value of  $I_x'$  is a minimum, and if  $d^2I_x'/d\alpha^2$  is negative when  $dI_x'/d\alpha = 0$ , the value of  $I_x'$  is a maximum. Therefore, of the two values of  $2\alpha$  determined by  $\tan 2\alpha = 2H/(I_y - I_x)$ , one value makes  $I_x'$  a maximum and the other a minimum. Since these values of  $2\alpha$  differ by  $180^\circ$ , the corresponding values of  $\alpha$  differ by  $90^\circ$ . This proves again that the principal axes are at right angles to each other.

The values of the moments of inertia with respect to the principal axes may be found by substituting in the expression for  $I_x'$  the values of  $\alpha$  determined by the equation

$$\tan 2\alpha = 2H/(I_y - I_x).$$

First changing  $I_x'$  to a function of  $2\alpha$ , we find

$$I_x' = \frac{1}{2}(I_x + I_y) - H \sin 2\alpha - \frac{1}{2}(I_y - I_x) \cos 2\alpha.$$

When

$$\tan 2\alpha = 2H/(I_y - I_x), \text{ we have}$$

$$\cos 2\alpha = \pm \frac{I_y - I_x}{[4H^2 + (I_y - I_x)^2]^{1/2}},$$

$$\sin 2\alpha = \pm \frac{2H}{[4H^2 + (I_y - I_x)^2]^{1/2}}.$$

Substituting these values in the expression for  $I_{x'}$ , it reduces to

$$I_{x'} = \frac{1}{2} \{ I_x + I_y \pm [4H^2 + (I_y - I_x)^2]^{1/2} \}.$$

The double sign corresponds to the two values of  $\alpha$  which make  $dI_{x'}/d\alpha = 0$ ; hence the minus sign gives the minimum value of  $I_{x'}$  and the plus sign the maximum value.

**68. Illustration.** To find the moments of inertia with respect to the principal axes through the center of the "Z" section of Fig. 165.

Choose the  $x$  and  $y$  axes parallel to the flanges and web respectively. Then

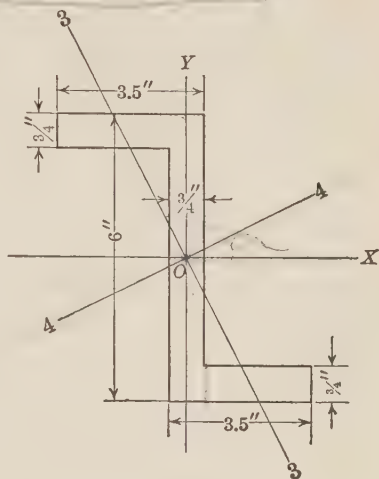


FIG. 165

$$I_x = 2 \left[ \frac{1}{3} (3.5)(3)^3 - \frac{1}{3} (2.75)(2.25)^3 \right] = 42.12,$$

$$I_y = \frac{1}{12} (6)(.75)^3 + 2 \left[ \frac{1}{12} (.75)(2.75)^3 + (.75)(2.75)(1.75)^2 \right] = 15.44,$$

$$H = \int_{-3/8}^{3/8} \int_{-3}^3 x y \, dx \, dy + \int_{-3\frac{1}{2}}^{-3\frac{3}{8}} \int_{2\frac{1}{4}}^3 x y \, dx \, dy + \int_{3/8}^{3\frac{1}{2}} \int_{-3}^{-2\frac{1}{4}} x y \, dx \, dy$$

$$= 0 - 9.475 - 9.475 = -18.95.$$

To determine the principal axes,  $\tan 2\alpha = 2H/(I_y - I_x) = 1.421$ , from which  $\alpha = 27^\circ 26'$  or  $117^\circ 26'$ . Inspection of the figure shows that the smaller angle in this case corresponds to the maximum moment of inertia.

The maximum and minimum moments of inertia are given by

$$\frac{1}{2} \{ I_x + I_y \pm [4H^2 + (I_y - I_x)^2]^{1/2} \} = 51.95 \text{ in.}^4 \quad \text{or} \quad 5.61 \text{ in.}^4$$

The area of the section is 8.63 sq. in. and hence the least radius of gyration is  $r_2 = \sqrt{5.61/8.63} = 0.81$  in.

## PROBLEMS

177. In a rectangle  $ABCD$ ,  $AB = 10''$ ,  $BC = 6''$ . Find the angles which the principal axes through  $A$  make with  $AB$ , and the values of the moments of inertia of the area of the rectangle with respect to these axes.

178. In a Z section the depth is  $5''$ , the flanges  $3.25''$ , and the thickness  $.5''$ . Find the position of the principal axes through the center, determine the maximum and minimum moments of inertia and the least radius of gyration.

**69. Product of Inertia referred to Principal Axes.** Having selected any rectangular axes through any point in a plane area, the principal axes of the area for that point are located with respect to the chosen axes by the formula  $\tan 2\alpha = 2H/(I_y - I_x)$ . If  $H = 0$ ,  $2\alpha = 0$  or  $180^\circ$ , and hence  $\alpha = 0$  or  $90^\circ$ , and therefore the chosen axes are themselves the principal axes for the point. Conversely, if the chosen axes are the principal axes,  $\alpha = 0$  and hence  $H = 0$ . That is, the product of inertia of an area referred to principal axes is zero, and conversely, any pair of axes for which the product of inertia is zero are principal axes.

Sometimes the principal axes can be found by inspection. As has just been shown, if  $H = 0$  the axes are principal axes. Now if either axis is an axis of symmetry of the area, the product of inertia of the area referred to the axes is zero; for, corresponding to any element of area with coordinates  $x$  and  $y$  there is an equal element of area with one coordinate the same and the other changed only in sign, so that the terms  $xy \, dA$ , of which  $H = \int xy \, dA$  is composed, occur in pairs which annul each other and thus reduce  $H$  to zero (Fig. 166). Of course it does not follow that conversely, if  $H = 0$ , one of the axes is an axis of symmetry.

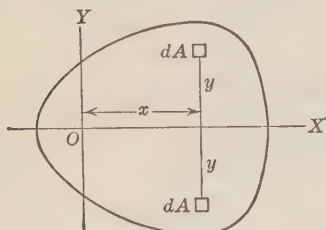


FIG. 166

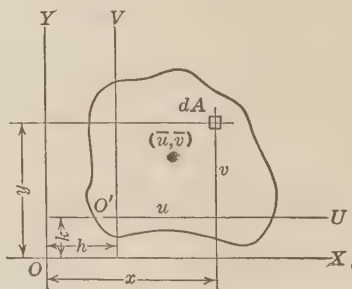


FIG. 167

**70. Products of Inertia of an Area referred to Parallel Axes.**

Let  $OX$ ,  $OY$ , and  $O'U$ ,  $O'V$  be two sets of parallel rectangular axes, the coordinates of  $O'$  being  $h$ ,  $k$  referred to  $OX$ ,  $OY$  (Fig. 167). Let  $H_{xy}$  and  $H_{uv}$  be the products of inertia of the area  $A$  referred to  $OX$ ,  $OY$ , and  $O'U$ ,  $O'V$ , respectively.

Since  $x = u + h$ ,  $y = v + k$ , we have

$$\begin{aligned} H_{xy} &= \int (u + h)(v + k) dA \\ &= \int uv dA + h \int v dA + k \int u dA + hk \int dA, \end{aligned}$$

or

$$H_{xy} = H_{uv} + h\bar{v}A + k\bar{u}A + hkA$$

where  $(\bar{u}, \bar{v})$  is the centroid of the area  $A$  referred to  $O'U$ ,  $O'V$ .

If  $O'$  is the centroid of  $A$ ,  $\bar{u} = \bar{v} = 0$ , and the equation becomes

$$H_{xy} = H_{uv} + hkA$$

If, in addition,  $O'U$  and  $O'V$  are principal axes of the area  $A$  through  $O'$ , the formula further reduces to

$$H_{xy} = hkA.$$

**71. Illustration.** As an illustration of the foregoing principles, the principal axes through the centroid

of the angle section 5" by 4" by 1/2" will be located and the moments of inertia of the area with respect to these axes computed.

Let  $OX$  and  $OY$  be the centroidal axes parallel to the legs, Fig. 168. Then

$$\bar{x} = \frac{(4)(0.5)(0.25) + (4.5)(0.5)(2.75)}{(4)(0.5) + (4.5)(0.5)} = 1.574,$$

$$\bar{y} = \frac{(4)(0.5)(2) + (4.5)(0.5)(0.25)}{(4)(0.5) + (4.5)(0.5)} = 1.074.$$

$$I_{ab} = (0.5)(4)^3/3 + (4.5)(0.5)^3/3 = 10.85,$$

$$I_{ac} = (0.5)(5)^3/3 + (3.5)(0.5)^3/3 = 20.98,$$

$$I_x = 10.85 - (4.25)(1.074)^2 = 5.95,$$

$$I_y = 20.98 - (4.25)(1.574)^2 = 10.45.$$

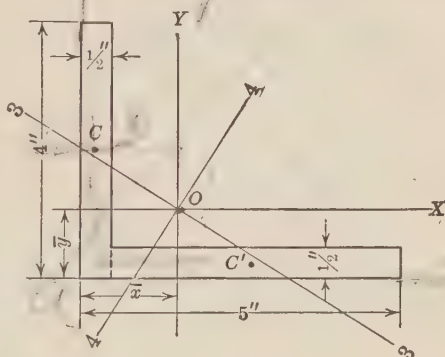


FIG. 168

The coordinates of the centers of the two rectangles of Fig. 168 referred to  $OX$  and  $OY$  are  $C(-1.324, 0.926)$  and  $C'(1.176, -0.824)$ . Hence, by the principle of § 70,

$$H = (-1.324)(0.926)(4)(0.5) + (1.176)(-0.824)(4.5)(0.5) = -4.632.$$

The directions of the principal axes are then given by the equation

$$\tan 2\alpha = 2H/(I_y - I_x) = -\frac{9.264}{4.50} = -2.059,$$

from which

$$\alpha = -32^\circ 3'.$$

The maximum and minimum moments of inertia of the area are given by

$$\begin{aligned} \frac{1}{2}[I_x + I_y \pm \{4H^2 + (I_y - I_x)^2\}^{1/2}] \\ = \frac{1}{2}(16.40 \pm 10.30) = 13.35 \text{ in.}^4 \quad \text{and} \quad 3.05 \text{ in.}^4 \end{aligned}$$

The least radius of gyration is

$$r_3 = \sqrt{3.05/4.25} = 0.85''.$$

The minimum moment of inertia is with respect to the axis 3-3 and the maximum with respect to the axis 4-4, Fig. 168.

## PROBLEMS

179. Locate the principal axes through the centroid of an angle section 8" by 6" by 1"; find the values of the maximum and minimum moments of inertia and the least radius of gyration.

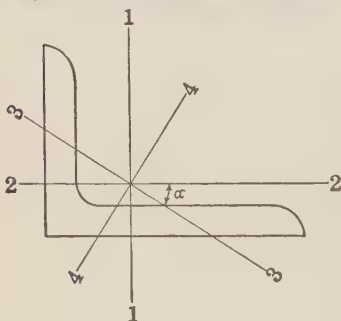


FIG. 169

180. The *Carnegie Pocket Companion* gives the following values for a certain angle section (Fig. 169): area = 13 sq. in.,  $I_1 = 80.8 \text{ in.}^4$ ,  $I_2 = 38.8 \text{ in.}^4$ , least radius of gyration (axis 3-3) = 1.28". From these data and the fact that

$$I_2 = I_3 \cos^2 \alpha + I_4 \sin^2 \alpha,$$

since axes 3-3 and 4-4 are principal axes, find the value of  $\alpha$ , the angle between 3-3 and 2-2.



**72. Relation between the Moments of Inertia of a Solid with respect to Parallel Axes.** In Fig. 170 let  $L$  and  $X$  be two parallel lines distant  $h$  apart. Let an origin of coordinates be chosen on  $X$  and take the  $y$  axis at right angles to  $L$  and  $X$  and in their plane. If  $dm$  is the mass of an element of a body of mass  $M$ , the moments of inertia of this body with respect to the lines  $L$  and  $X$  are  $I_l = \int r^2 dm$  and  $I_x = \int r'^2 dm$  respectively, where  $r$  and  $r'$

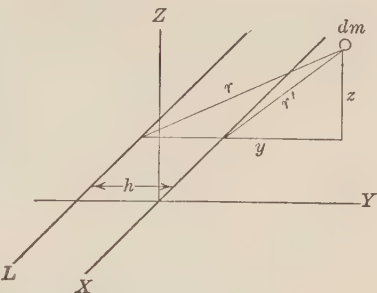


FIG. 170

respectively. If  $y$  and  $z$  are the coordinate of  $dm$  with respect to the planes  $xz$  and  $xy$  respectively, then

$$\begin{aligned} I_l &= \int r^2 dm = \int [(h + y)^2 + z^2] dm \\ &= h^2 \int dm + 2h \int y dm + \int (y^2 + z^2) dm \\ &= h^2 M + 2h \bar{y} M + I_x. \end{aligned}$$

If  $X$  passes through the centroid of the body,  $\bar{y} = 0$  and, if we call the moment of inertia of the body with respect to the centroidal axis  $I_c$ , the formula becomes

$$I_l = I_c + h^2 M.$$

### PROBLEMS

181. In § 60 it was found that the moment of inertia of a sphere with respect to a diameter is  $2Mr^2/5$ . From this result prove that the moment of inertia of a sphere with respect to a tangent line is  $7Mr^2/5$ .

182. In Problem 150 it was found that the moment of inertia of a rectangular solid with respect to a centroidal line perpendicular to the face of sides  $a$  and  $b$  is  $M(a^2 + b^2)/12$ . From this result find the moment of inertia of the solid with respect to an edge parallel to the line mentioned.

**73. Moment of Inertia of a Solid by Parallel Sections.** When a body can be cut by parallel planes in similar sections, it is often convenient to so divide the body in finding its moment of inertia with respect to a line either perpendicular to or parallel to the

sections. Consider any disk of small thickness  $t$  and let  $AB$  be any line parallel to and lying between the two plane faces, Fig. 171. Let the mass per unit volume be  $m$ . Then the mass of an element of volume of base  $dA$  and thickness  $t$  is  $mt dA$  and if  $x$  is the distance parallel to the face of the disk from this element to

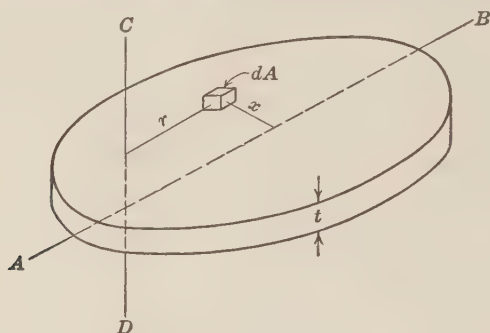


FIG. 171

the line  $AB$ , the moment of inertia of the disk with respect to  $AB$  is approximately  $\int m t x^2 dA$ , if  $t$  is very small. Since  $m$  and  $t$  are constant, this may be written  $mt \int x^2 dA$ , and since  $\int x^2 dA$  represents the moment of inertia of the area of a section of the disk parallel to its plane faces, with respect to  $AB$ , the result may be stated as follows.

*The moment of inertia of a thin disk with respect to a line parallel to and lying between its plane faces is approximately equal to the mass per unit volume, times the thickness, times the moment of inertia, with respect to the line, of a section of the disk parallel to its faces and containing the given line.*

In the same way, the moment of inertia of the disk with respect to a line  $CD$ , perpendicular to the faces of the disk, is  $mt \int r^2 dm$ , Fig. 171. This result may be stated as follows.

*The moment of inertia of a thin disk with respect to a line perpendicular to its plane faces is approximately equal to the mass per unit volume, times the thickness, times the polar moment of inertia of the face of the disk with respect to the given line.*

In applying these theorems to solids the disks used will be of infinitesimal thickness, so that in finding the moment of inertia of the body a limit of a sum will be taken in which each term

represents approximately the moment of inertia of a disk whose thickness approaches zero as a limit. The limit of the sum of these terms then gives an exact value for the moment of inertia of the body.

**74. Illustration.** A solid of mass  $m$  per unit volume is enclosed by the surface generated by revolving a parabola about its axis and a plane perpendicular to its axis, Fig. 172. It is required to find the moment of inertia of this solid, (a) with respect to the axis of the parabola, (b) with respect to a line through the vertex perpendicular to the axis of the parabola.

(a) Let the solid be divided into disks of thickness  $dx$  and radius  $y$  by planes perpendicular to the axis of the parabola. Then the moment of inertia of any one of the disks is approximately equal to  $m dx \pi y^4/2$ , since the face of the disk is a circle of radius  $y$ , for which the polar moment of inertia with respect to the center is  $\pi y^4/2$ . The moment of inertia of the whole solid with respect to  $OX$  is then

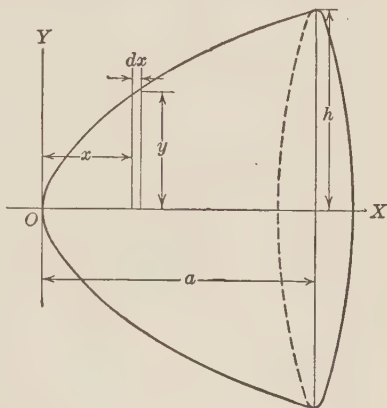


FIG. 172

$$I_x = \frac{m\pi}{2} \int_0^a y^4 dx.$$

But from the equation of the parabola,  $y^2 = h^2 x/a$ , and therefore

$$I_x = \frac{m\pi h^4}{2a^2} \int_0^a x^2 dx = \frac{m\pi h^4 a}{6}.$$

If  $M$  is the mass of the solid,

$$M = m\pi \int_0^a y^2 dx = \frac{m\pi h^2}{a} \int_0^a x dx = \frac{m\pi h^2 a}{2}.$$

Therefore

$$I_x = Mh^2/3.$$

(b) If  $x$  is the abscissa of the centroid of any disk and  $y$  is the corresponding ordinate of the parabola, the moment of inertia of the

disk with respect to a diameter along  $y$  is  $m dx \pi y^4/4$ , and the moment of inertia of this disk with respect to the  $y$  axis is therefore

$$m dx \pi y^4/4 + (m dx \pi y^2)x^2,$$

and the moment of inertia of the whole body with respect to the  $y$  axis is then

$$\begin{aligned} I_y &= m\pi \int_0^a \left( \frac{y^4}{4} + y^2 x^2 \right) dx = m\pi \int_0^a \left( \frac{h^4 x^2}{4a^2} + \frac{h^2 x^3}{a} \right) dx \\ &= m\pi \left( \frac{h^4 a}{12} + \frac{h^2 a^3}{4} \right) = \frac{M}{6} (h^2 + 3a^2). \end{aligned}$$

### PROBLEMS

183. Prove that the moment of inertia of a right circular cylinder of radius  $r$ , altitude  $h$ , and mass  $M$  with respect to the axis of the cylinder is  $Mr^2/2$  and with respect to a diameter of one end of the cylinder is  $M(3r^2 + 4h^2)/12$ . From this result show that the moment of inertia of the cylinder with respect to a line that passes through the centroid of the cylinder perpendicular to its axis is  $M(3r^2 + h^2)/12$ . To what forms do the last two expressions reduce when  $r$  is negligible compared with  $h$ , that is, when the cylinder becomes a slender rod? Derive the formulas for the slender rod directly.

184. Prove that the moment of inertia of a right circular cone of radius  $r$ , altitude  $h$ , and mass  $M$  with respect to the axis of the cone is  $3Mr^2/10$ , and with respect to a line through the vertex perpendicular to the axis of the cone is  $3M(r^2 + 4h^2)/20$ . Show also that the moment of inertia of the cone with respect to a line through the centroid parallel to the base is  $3M(4r^2 + h^2)/80$ .

185. A rectangular solid has edges  $a$ ,  $b$ , and  $c$ , and mass  $M$ . By making sections parallel to the face of sides  $b$  and  $c$ , find the moment of inertia of the body with respect to a centroidal axis parallel to  $c$ . Using the same sections, find the moment of inertia of the body with respect to a line through the center of the face of one end parallel to one side of the face.

186. A right pyramid has a rectangular base of sides  $a$  and  $b$ , an altitude  $h$  and a mass  $M$ . Find its moment of inertia with respect to a line through the vertex parallel to the side  $b$  of the base. Then find the moment of inertia with respect to a centroidal axis parallel to the edge  $b$ , and with respect to a line parallel to  $b$  through the center of the base.

187. The ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  is revolved around the  $x$  axis. Find the moment of inertia of the spheroid enclosed with respect to the  $y$  axis; with respect to the  $x$  axis.

188. A wedge-shaped solid is of length  $b$  and a section at right angles to its length is an isosceles triangle of base  $a$  and altitude  $c$ . Find the moment of inertia of the wedge with respect to its edge.

189. A right cylinder of height  $h$  and mass  $M$  has an ellipse of semi-axes  $a$  and  $b$  as base. Prove that the moment of inertia of the cylinder with respect to a line containing the diameter  $2a$  of the base is  $M(3b^2 + 4h^2)/12$ . Find also the moment of inertia of the cylinder with respect to a centroidal axis parallel to the diameter  $2a$ .

**75. The Ellipse of Inertia.** It was shown in § 67 that for any plane area a pair of rectangular axes,  $OX$  and  $OY$ , can be found through any point in the plane such that the product of inertia of the area referred to these axes is zero, and that this pair of axes are principal axes of the area for this point. If  $OU$ , Fig. 173, is a line of the plane passing through the origin and making an angle  $\alpha$  with  $OX$ , one of the principal axes through  $O$ , then

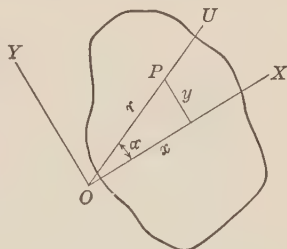


FIG. 173

$$I_u = I_x \cos^2 \alpha + I_y \sin^2 \alpha,$$

since the product of inertia referred to principal axes is zero. Replacing  $I$  by  $Ak^2$ , we may write this equation in the form

$$1 = \frac{k_x^2 \cos^2 \alpha}{k_u^2} + \frac{k_y^2 \sin^2 \alpha}{k_u^2}.$$

In this equation  $k_u$  and  $\alpha$  are variables. On  $OU$  take a point  $P$  at a distance  $r$  from  $O$  such that  $r = OP = 1/k_u$  and let  $k_x = 1/a$ ,  $k_y = 1/b$ . The equation becomes

$$\frac{r^2 \cos^2 \alpha}{a^2} + \frac{r^2 \sin^2 \alpha}{b^2} = 1.$$

The coordinates of  $P$  referred to the principal axes are  $x = r \cos \alpha$ ,  $y = r \sin \alpha$ , and therefore the equation may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the point  $P$  traces an ellipse as  $\alpha$  goes through all values from 0 to  $2\pi$ . This ellipse is called the *ellipse of inertia* of the area for the given point. Its semi-axes are the reciprocals of the radii of gyration of the area with respect to the principal axes and are located on those axes. The distance from the center  $O$  to any point  $P$  on the ellipse of inertia is the reciprocal of the radius of gyration of the area with respect to the line  $OP$ .



76. Illustration. The "Z" section used in the illustration of § 68 was found to have principal axes inclined at  $27^\circ 26'$  and  $117^\circ 26'$  with the axis parallel to the flanges and the values of the moments of inertia of the section with respect to these principal axes were found to be

$$I_4 = 51.95 \text{ in.}^4,$$

$$I_3 = 5.61 \text{ in.}^4$$

The area is  $8.63 \text{ in.}^2$  and hence the semi-axes of the ellipse of inertia are, since

$$a = 1/k_4, \quad b = 1/k_3,$$

$$a = \sqrt{8.63/51.95} = 0.408'',$$

$$b = \sqrt{8.63/5.61} = 1.24''.$$

The ellipse of inertia is sketched in Fig. 174, using a different scale from that used in the construction of the "Z" section.

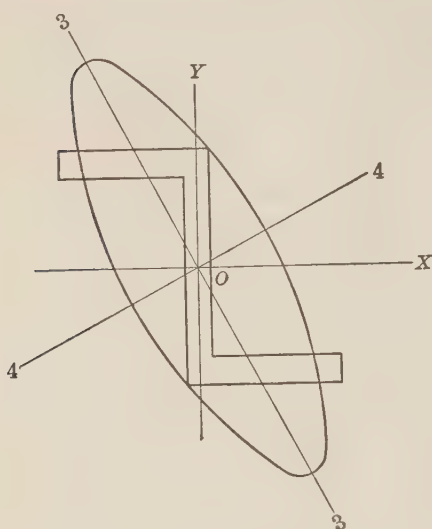


FIG. 174

Of course the ellipse of inertia does not give any information that is not contained in the equation  $I_u = I_x \cos^2 \alpha + I_y \sin^2 \alpha$ . It simply shows graphically the relation between the radii of gyration with respect to all lines through the chosen point.

### PROBLEMS

190. Sketch carefully the ellipse of inertia of the Z section used in the preceding illustration, taking the values of the moments of inertia with respect to the principal axes and the position of the principal axes as found in § 68. By scaling the intercepts of the ellipse on the axes parallel and perpendicular to the flanges, compute the moments of inertia with respect to these lines and compare the values with those found in § 68. Also compute these values by substituting the proper values of  $\alpha$  in  $I_u = I_x \cos^2 \alpha + I_y \sin^2 \alpha$ .

191. Sketch the ellipse of inertia of the area of a rectangle of dimensions  $10''$  by  $6''$  with respect to the center; with respect to the center of a  $6''$  side; with respect to a vertex.

192. Sketch the ellipse of inertia of the area of a square of side  $a$  with respect to the center of the square; with respect to a vertex.

193. Sketch the ellipse of inertia of the area of an ellipse with respect to the center; with respect to the one end of the major axis.



**77. Moment of Inertia of a Solid with respect to an Inclined Axis.** Let  $OU$  make with the coordinate axes,  $OX$ ,  $OY$ ,  $OZ$ , angles  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively and let  $l = \cos \alpha$ ,  $m = \cos \beta$ ,  $n = \cos \gamma$ , Fig. 175. If  $dm$  is an element of mass of a body of mass  $M$ , located at  $(x, y, z)$  distant  $p$  from  $OU$ , then the moment of inertia of the body with respect to  $OU$  is  $I_u = \int p^2 dm$ .

Now  $p^2 = OP^2 - OM^2$  and  $OM$  is the sum of the projections of  $x$ ,  $y$ , and  $z$  on  $OU$ . Therefore

$$OM = x \cos \alpha + y \cos \beta + z \cos \gamma,$$

or

$$OM = lx + my + nz,$$

hence

$$\begin{aligned} p^2 &= (x^2 + y^2 + z^2) - (lx + my + nz)^2 \\ &= x^2(1 - l^2) + y^2(1 - m^2) + z^2(1 - n^2) \\ &\quad - 2lmxy - 2lnxz - 2mnyz. \end{aligned}$$

But  $l^2 + m^2 + n^2 = 1$ ; hence

$$1 - l^2 = m^2 + n^2, \quad 1 - m^2 = l^2 + n^2, \quad 1 - n^2 = l^2 + m^2,$$

and therefore

$$\begin{aligned} p^2 &= l^2(y^2 + z^2) + m^2(x^2 + z^2) + n^2(x^2 + y^2) \\ &\quad - 2lmxy - 2lnxz - 2mnyz, \end{aligned}$$

and

$$\begin{aligned} I_u &= l^2 \int (y^2 + z^2) dm + m^2 \int (x^2 + z^2) dm + n^2 \int (x^2 + y^2) dm \\ &\quad - 2lm \int xy dm - 2ln \int xz dm - 2mn \int yz dm. \end{aligned}$$

Since

$$I_x = \int (y^2 + z^2) dm, \quad I_y = \int (x^2 + z^2) dm, \quad I_z = \int (x^2 + y^2) dm,$$

the value of  $I_u$  becomes

$$\begin{aligned} I_u &= l^2 I_x + m^2 I_y + n^2 I_z \\ &\quad - 2lm \int xy dm - 2ln \int xz dm - 2mn \int yz dm. \end{aligned}$$

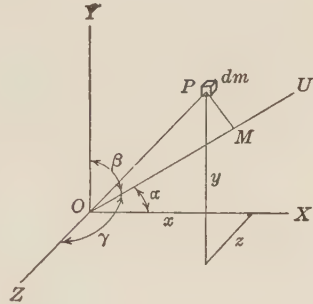


FIG. 175

If two of the coordinate planes are planes of symmetry, all the products of inertia reduce to zero and the equation takes the form

$$I_u = l^2 I_x + m^2 I_y + n^2 I_z.$$

If the products of inertia referred to the chosen axes do not vanish, it can be shown that axes do exist for any point referred to which the products of inertia do vanish. The proof will not be given here, as it is somewhat beyond the scope of this course.

**78. Principal Axes of a Solid; The Ellipsoid of Inertia.** The axes through any point for which the three products of inertia of a body vanish are called the principal axes of the body for that point. The expression for the moment of inertia of the body with respect to a line  $OU$  is then

$$I_u = l^2 I_x + m^2 I_y + n^2 I_z,$$

where  $l, m, n$  are the direction cosines of the line  $OU$  referred to the principal axes  $OX, OY, OZ$  through  $O$ .

Replacing  $I$  by  $Mk^2$  in each case and letting  $k_x = 1/a, k_y = 1/b, k_z = 1/c, k_u = 1/r$ , this equation reduces to

$$\frac{l^2 r^2}{a^2} + \frac{m^2 r^2}{b^2} + \frac{n^2 r^2}{c^2} = 1.$$

If  $P$  is taken on  $OU$  such that  $OP = r = 1/k_u$ , the coordinates of  $P$  are  $x = lr, y = mr, z = nr$  and the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of an ellipsoid.

This ellipsoid is called the *ellipsoid of inertia* of the body with respect to the chosen point. It is a surface such that the distance from its center to a point on its surface measured along any line is the reciprocal of the radius of gyration of the body with respect to that line.

## PROBLEMS

194. Find the semi-axes of the ellipsoid of inertia of a rectangular block of dimensions 10" by 8" by 6" with respect to the center of the block and show how they are located.

195. Find the semi-axes of the ellipsoid of inertia of a right circular cone with radius of base 10" and height 20" with respect to the vertex.

**79. Graphical Method of Determining the Moment of Inertia of a Plane Area.** In computing the moment of inertia of a plane area with respect to a line in its plane, the area of each strip of infinitesimal width parallel to the given line is multiplied by the square of its distance from the line. By a geometric construction it is possible so to change the strips that the moment of inertia may be obtained by multiplying all the areas of the strips by the square of the same distance. The problem is thus reduced to that of finding the area made up of the changed strips. The method is as follows.

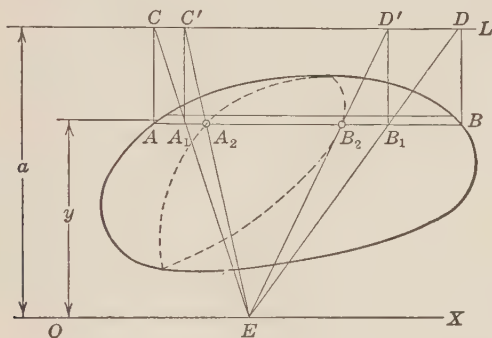


FIG. 176

In Fig. 176 it is desired to find the moment of inertia with respect to  $OX$  of the area bounded by the full line curve. Any chord  $AB$ , at a distance  $y$  from  $OX$ , is projected upon a line  $L$  at a distance  $a$  from  $OX$ . The ends,  $C$  and  $D$ , of the projection are joined to any point  $E$  of  $OX$  by lines cutting  $AB$  in  $A_1$  and  $B_1$ . The points  $A_1$  and  $B_1$  are projected upon  $L$  and their projections,  $C_1$  and  $D_1$ , are joined to  $E$  by lines cutting  $AB$  in  $A_2$  and  $B_2$ . By similar triangles, we have

$$\frac{A_2B_2}{C_1D_1} = \frac{y}{a}, \quad \text{and} \quad \frac{A_1B_1}{CD} = \frac{y}{a}.$$

Then, noting that  $A_1B_1 = C_1D_1$  and  $AB = CD$ , we find, by multiplication of corresponding terms of the equations,

$$\frac{A_2B_2}{AB} = \frac{y^2}{a^2},$$

or

$$y^2 AB = a^2 A_2 B_2.$$

Hence the moment of inertia of the given area becomes

$$I_x = \int y^2 AB \, dy = a^2 \int A_2 B_2 \, dy.$$

This latter expression is  $a^2$  times the area enclosed by the curve traced out by  $A_2$  and  $B_2$  as the chord  $AB$  moves through all positions from the nearest chord to  $OX$  to the one farthest away. Hence if the area of this auxiliary curve traced out by  $A_2$  and  $B_2$  is found, the moment of inertia with respect to  $OX$  of the original area is obtained by multiplying this area by  $a^2$ .

The area may be obtained approximately by the use of a planimeter or by one of several approximate methods. If, for example, the area is to be determined by Simpson's Rule, the distance between the tangents to the boundary of the area parallel to  $OX$ , nearest to and farthest from  $OX$ , is divided into an even number of equal parts and chords  $AB$  parallel to  $OX$  drawn through the points of division. The value of  $A_2 B_2$  can then be scaled and substituted in the formula for the area.

### PROBLEMS

196. An oval-shaped figure lies between and is tangent to two parallel lines,  $L_1$  and  $L_2$ , 10 inches apart. The chords parallel to  $L_1$  and  $L_2$  at intervals of 1 inch, counting from  $L_1$ , are 5.2, 6.8, 7.4, 8.3, 8.0, 7.6, 6.6, 5.8, 4.3 inches. Find the approximate moment of inertia of the area with respect to a line 2 inches from  $L_1$  and 12 inches from  $L_2$ .

197. Using the graphical method described above, find the moment of inertia of the area of a circle about a diameter, (a) when 5 chords are used, (b) when 9 chords are used. Compare with the exact value  $\pi r^4/4$ .

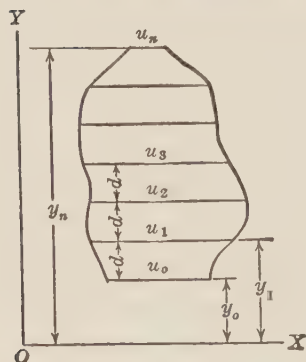


FIG. 177

80. Moment of Inertia of a Plane Area by Simpson's Rule. In Fig. 177 is shown an area divided into an even number,  $n$ , of strips of equal width, parallel to the axis with respect to which the moment of inertia of the area is desired, in this case the  $x$  axis.

If  $u_0, u_1, u_2, \dots, u_n$  are the lengths of the bases of the strips,  $y_0, y_1, y_2, \dots, y_n$  the distances from the  $x$  axis to these bases, and  $d = (y_n - y_0)/n$ , the expression for an approximate value of the moment of inertia of the area with respect to the  $x$  axis is

$$\frac{d}{3} [y_0^2 u_0 + 4(y_1^2 u_1 + y_2^2 u_2 + \dots + y_{n-1}^2 u_{n-1}) + 2(y_2^2 u_2 + y_4^2 u_4 + \dots + y_{n-2}^2 u_{n-2}) + y_n^2 u_n].$$

This follows from the fact that if  $u$  is the breadth of the area at any distance  $y$  from  $OX$ , the exact expression for the moment of inertia of the area with respect to  $OX$  is

$$I_x = \int_{y_0}^{y_n} y^2 u dy,$$

and by Simpson's Rule, § 57, this integral is approximately equal to the expression written above.

**81. Moment of Inertia by Direct Addition.** An approximate value for the moment of inertia of an area with respect to a line in its plane may be obtained by dividing the area into narrow strips parallel to the given line and multiplying the area of each strip by the square of the distance of the middle line of each strip from the axis and adding all such products together.

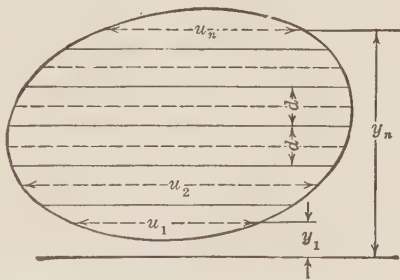


FIG. 178

This follows at once from the definition of moment of inertia. Thus in Fig. 178, if the strips are all of width  $d$  and the middle lines of the strips are of length  $u_1, u_2, \dots, u_n$ , distant respectively  $y_1, y_2, \dots, y_n$  from  $OX$ , the value of the moment of inertia of the area with respect to  $OX$  is approximately

$$I_x = d(u_1 y_1^2 + u_2 y_2^2 + \dots + u_n y_n^2).$$

## PROBLEMS

198. Compute the moment of inertia of the area bounded by the curve  $y^2 = 4x$ , the  $x$  axis and the line  $x = 9$  with respect to the  $x$  axis, by Simpson's Rule, dividing the area into six strips of equal width and obtaining the lengths of the bases of these strips from the equation of the curve. Compute the exact value by integration and find the percentage of error when the approximate method is used. Also make the computation by Simpson's Rule using ten strips.

199. Compute the moment of inertia of the area described in Problem 198 by the method of direct addition, using six strips, also using ten strips. Obtain the lengths of the center lines of the strips from the equation of the curve.

200. Solve Problem 196: (a) by the method of direct addition; (b) by the use of Simpson's Rule.



## CHAPTER V

### WEIGHTED CORDS AND LINKS, PARABOLA, CATENARY

**82. Cords Carrying Concentrated Loads.** In Fig. 179, weights  $W_1, W_2, W_3$  are attached to four cords of lengths  $l_1, l_2, l_3, l_4$  attached to two points  $A$  and  $B$ , the cords being assumed to be of negligible weight and inextensible. The cords may then be assumed to be straight.

At each point of attachment of a weight three forces act, the weight and the tensions in the cords attached there. For the equilibrium of the forces acting at the point, two equations may be written, involving the tensions in the cords attached there

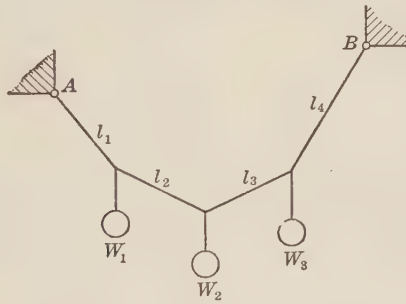


FIG. 179

and the angles of inclination of these cords with the horizontal. If there are  $n$  weights, there are  $n + 1$  cords. Since the tension and inclination of each cord is unknown, there are  $2n + 2$  unknowns. The two equations expressing the equilibrium of the forces at each point of attachment make  $2n$  equations in all and an additional two equations may be obtained by projecting the lengths of the cords on the horizontal and vertical lines between the points of support. Thus a total of  $2n + 2$  equations can be obtained, from which, theoretically, it is possible to obtain the values of the tensions in the cords and their angles of inclination. However, if more than a single weight is thus suspended by

strings of given lengths, the elimination of the angles of inclination of the strings from the equations of equilibrium leads to equations of higher than second degree in the tensions, and the solutions can be obtained only by some method of approximation. If, on the other hand, the lengths of the strings are not given, but the weights are restricted to act in definite vertical lines, the lengths of the cords and the tensions can be determined easily either by algebraic or graphical methods. These methods will be discussed in the next paragraphs.

### PROBLEMS

201. Two cords of length 10 ft. and 6 ft. are attached at the same point to a weight  $W$ . The other ends of the cords are attached to two supports  $A$  and  $B$  situated as shown in Fig. 180. Find the tensions in the cords. Solve graphically and algebraically.

(SUGGESTION. To solve algebraically, find the angles which the strings make with the horizontal and use Lami's theorem.)

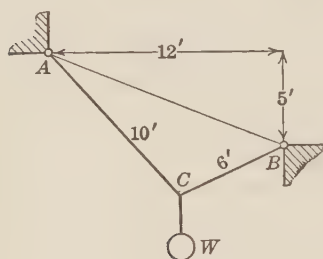


FIG. 180

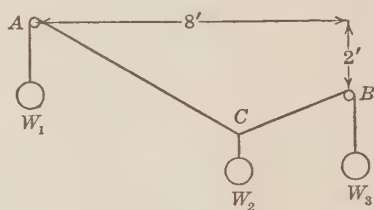


FIG. 181

202. In Fig. 181,  $A$  and  $B$  are two small smooth pegs over which passes the cord carrying the weights shown, the weight  $W_2$  being knotted to the cord at  $C$ . If the weights are in equilibrium when  $W_1 = 400$  lbs.,  $W_2 = 500$  lbs.,  $W_3 = 300$  lbs., find the angles of inclination of the segments of the cord and then determine the horizontal and vertical distances of  $C$  from  $A$ . Solve algebraically and check by graphical construction. Under what conditions would equilibrium of the weights be impossible? Could there be equilibrium if  $W_1 = 100$  lbs.,  $W_2 = 320$  lbs.,  $W_3 = 200$  lbs.?

203. A weight  $W$  is suspended by two strings of lengths 7 ft. and 4 ft. attached to the weight at the same point, the other ends being attached to supports  $A$  and  $B$ , where  $A$  is 6 ft. higher than  $B$  and 8 ft. to the right of  $B$ . Find the tensions in the cords. Solve algebraically and check by graphical construction.

204. In Fig. 181 what relation exists between the weights when  $C$  is on the same level as  $B$ ? When  $C$  is higher than  $B$ ? Lower than  $B$ ?

### 83. Suspended Weights Hanging in Assigned Vertical Lines.

Figure 182 represents weights  $W_1$ ,  $W_2$ ,  $W_3$ , suspended by a cord attached to two supports  $A$  and  $B$  and restricted to hang in vertical lines distant  $a_1$ ,  $a_2$ , and  $a_3$  from  $A$ . Evidently if the cord, when in equilibrium, should become rigid, the system of forces would still be in equilibrium. The weights and reactions of the supports may then be regarded as a set of forces in equilibrium, and the conditions of equilibrium may be applied.

Hence the horizontal components of the reactions of the supports at  $A$  and  $B$  are equal, and the sum of the vertical components of these reactions is equal to the sum of the weights of the suspended bodies. The only other independent condition for equilibrium is that the sum of the moments of the forces with respect to any point is equal

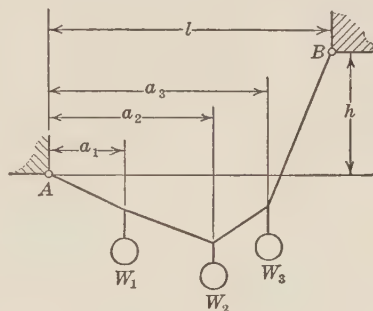


FIG. 182

to zero. Since there are four unknowns, the components of the reactions of the supports, and only three independent equations for their determination, it follows that some other condition must be imposed in order that the solution be determinate. This is also evident from the consideration that lengthening or shortening the cord will evidently change the values of the reactions of the supports.

Suppose then that we have given not only the weights, the lines in which they hang, and the points of support, but also the value of the horizontal component of the reactions of the supports. We may then proceed to find the shape of the polygon which the cord assumes, the lengths of the segments, and the tensions in the segments. The method of solution is indicated in the following illustration.

**84. Illustration.** Suppose in Fig. 183,  $W_1 = 80$  lbs.,  $W_2 = 100$  lbs.,  $W_3 = 50$  lbs.,  $l = 10$  ft.,  $h = 2$  ft.,  $a_1 = 3$  ft.,  $a_2 = 6$  ft.,  $a_3 = 8$  ft., and that the length of the cord is so chosen that the horizontal com-

ponent,  $H$ , of the reactions of the supports is 40 lbs. To find the tensions in the segments of the cord and the lengths of the segments.

Taking moments about  $A$ , we have

$$10Y_2 = (80)(3) + (100)(6) + (50)(8) + (40)(2),$$

or

$$Y_2 = 132 \text{ lbs.}$$

Then

$$Y_1 = 98 \text{ lbs.}$$

If we represent the tensions in the segments, beginning at  $A$ , by  $T_1, T_2$ , etc., the value of  $T_1$  is  $(40^2 + 98^2)^{1/2} = 105.8$  lbs.

To find  $T_2$ , note that the components of  $T_1$  are 40 and 98 and that therefore

$$T_2 = (40^2 + 18^2)^{1/2} = 43.86 \text{ lbs.}$$

In a similar manner, we find

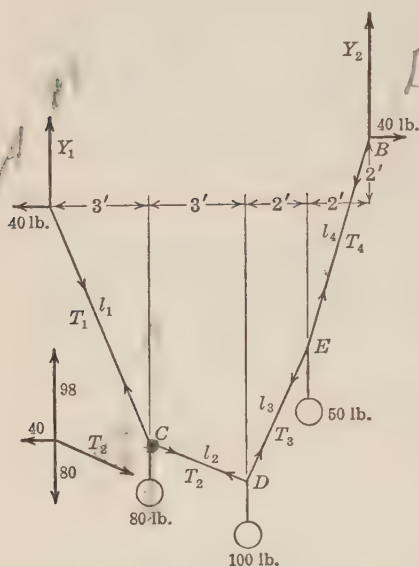


FIG. 183

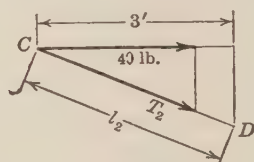


FIG. 184

$$T_3 = (40^2 + 82^2)^{1/2} = 91.24 \text{ lbs.,}$$

$$T_4 = (40^2 + 132^2)^{1/2} = 137.9 \text{ lbs.}$$

The lengths of the segments are easily obtained by comparison of similar triangles at the points of attachment of the weights. For example, at  $C$ , letting  $CD = l_2$ ,  $l_2/3 = T_2/40 = 43.86/40$  or  $l_2 = 3.29$  ft. (Fig. 184). Similarly,  $l_1/3 = T_1/40$ , or  $l_1 = 3(105.8)/40 = 7.94$  ft.,  $l_3 = 2(91.24)/40 = 4.56$  ft.,  $l_4 = 2(137.9)/40 = 6.90$  ft.

### PROBLEMS

205. Weights of 24 lbs., 40 lbs., and 16 lbs. are suspended by a cord attached to two supports  $A$  and  $B$ , where  $B$  lies 8 ft. to the right of and 4 ft. lower than  $A$ , and they hang in vertical lines distant 2 ft., 5 ft., and 7 ft. from  $A$ . If the horizontal component of the tension in any segment of the cord is 32 lbs., find the tensions in the segments of the cord and the lengths of the segments.

206. In the preceding problem, instead of having given the horizontal component of the tension, let the maximum tension that occurs in the cord be 100 lbs. Find the remaining tensions and the lengths of the segments.

207. In Problem 205, instead of having given the horizontal component of

the tension, choose this component so that the last segment of the string on the right is horizontal. Find the tensions in the segments and the lengths of the segments.

208. Two weights of 1000 lbs. and 500 lbs. are attached to cords and suspended from two supports  $A$  and  $B$ , where  $B$  is 20 ft. to the right of and 4 ft. lower than  $A$ . What lengths of cord are necessary if the 1000-lb. weight is to hang in a vertical line 6 ft. from  $A$ , the 500-lb. in a vertical line 6 ft. from  $B$ , and the segment between the weights is to be horizontal? What are the tensions in the segments? Exchange the weights and solve.

**85. Graphical Method of Solution.** Figure 185 (a) represents a cord suspended from two supports and carrying weights  $W_1, W_2, W_3$  attached to the cord at definite points on it. Since if the cord were to become a rigid body the forces acting on it would still hold it in equilibrium, the weights and the reactions of the supports form a system of forces in equilibrium. If these forces

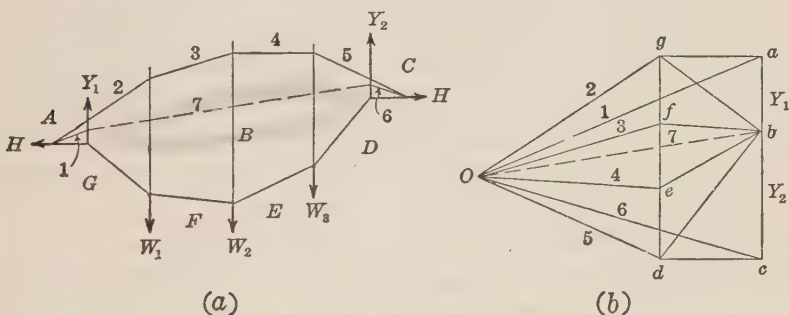


FIG. 185

are laid off in succession, they form a closed force polygon. If a point  $O$  is chosen anywhere in the plane of the forces, and lines are drawn from  $O$  to the vertices of the force polygon, a corresponding equilibrium polygon may be drawn with sides parallel to the lines drawn from  $O$  and with vertices on the corresponding lines of the forces, as explained in § 29. If then the points of support of the cord, the values of the weights, the lines in which the weights are to hang and the horizontal components of the tensions in the cord are given, the values of the vertical components of the reactions of the supports may be obtained graphically by laying out to scale the force polygon  $agfedca$ , Fig. 185 (b), in which the side  $ca$  is the sum of the vertical components,  $Y_2$  and  $Y_1$ , of the reactions of the supports.



Then, if we assume any point  $O$  and draw lines from  $O$  to the known vertices of the force polygon,  $a, g, f, e, d, c$ , and the corresponding parallel lines to points on the lines of the forces, the point  $b$ , which separates  $Y_2$  and  $Y_1$ , is determined by drawing  $Ob$  parallel to the side "7" of the equilibrium polygon. If the point  $b$  is then connected with the other points of the force polygon,  $g, f, e, d$ , the lines  $gb, fb, eb, db$  represent the magnitudes and directions of the forces in the corresponding segments,  $GB, FB, EB, DB$ , of the cord. For at the left support, the three forces,  $Y_1, H$  and the tension  $T_1$  in the cord are in equilibrium, and since  $ba$  and  $ag$  represent two of the forces, the third side,  $gb$ , of the triangle must represent the third force,  $T_1$ . In the same way,  $bg$  and  $gf$  represent two of the three forces acting at the point of attachment of the weight  $W_1$ , and hence  $fb$  must represent the magnitude and direction of the third force,  $T_2$ , acting at the point. From this construction the shape of the polygon assumed by the cord is determined by drawing lines  $BG, BF$ , etc., parallel to the corresponding lines  $gb, fb$ , etc. The values of the tensions in the different segments and the lengths of the segments can then be scaled from the figure.

The polygon assumed by the string is an equilibrium polygon corresponding to the force polygon  $ag \cdots ba$ . This polygon will be referred as the *string polygon* and the point  $b$  as its *pole*.

### PROBLEMS

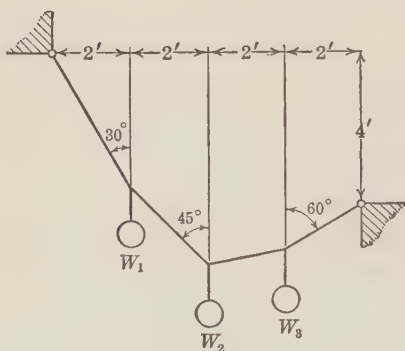


FIG. 186

209. Solve Problem 205 graphically.

210. Weights of 100 lbs., 120 lbs., 150 lbs., and 200 lbs. are to be suspended by a cord attached to points  $A$  and  $B$ , where  $B$  lies 15 ft. to the right of and 3 ft. above  $A$ . If the weights are to hang in vertical lines distant 3 ft., 6 ft., 8 ft., and 12 ft. from  $A$ , in the order named above, and the horizontal component of the tension is to be 300 lbs., determine graphically the shape of the string polygon, the lengths of

the segments, and the tensions in the segments.



211. Five equal weights are to be suspended from a cord supported at two points on the same level and are to hang in lines equally spaced between the points of support. If the maximum tension in the cord is to be three times the value of one of the weights, construct graphically the shape assumed by the cord, and find the total length of the cord and the least tension in any of the segments. Check algebraically.

212. Eight equal weights are to be hung on a cord supported at two points on the same level and are to hang in lines equally spaced between the points of support. If the horizontal component of the tension in the cord is to be equal to four times the value of one weight, draw the shape of the string polygon and determine the length of the cord and the maximum tension. Check algebraically.

213. In Fig. 186, the weights hang as shown. By graphical construction find the ratios  $W_1 : W_2 : W_3$ .

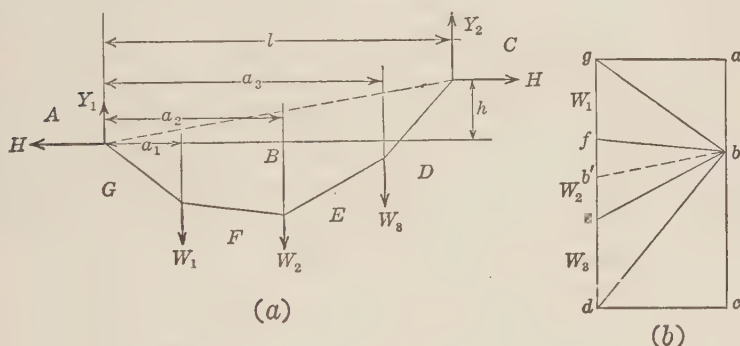


FIG. 187

86. **Locus of the Pole of the String Polygon.** Using the notation shown in Fig. 187, we have by taking moments about the left support

$$Y_2 = (a_1W_1 + a_2W_2 + a_3W_3)/l + hH/l.$$

Let

$$Y_2' = (a_1W_1 + a_2W_2 + a_3W_3)/l.$$

Then

$$Y_2 = Y_2' + hH/l.$$

If either  $h = 0$  or  $H = 0$ ,  $Y_2$  becomes  $Y_2'$  and therefore  $Y_2'$  is the value that  $Y_2$  would have if either the points of support were on the same level, or if the cord were replaced by a rigid body supported by two smooth pins at the points of support, with no horizontal force at the pins.

The value of  $Y_2'$  may be computed by moments and laid off vertically from  $d$ , or it may be determined graphically by constructing the equilibrium polygon for vertical forces only, assuming  $H$  to be zero. If  $db' = Y_2'$  and  $Y_2 = cb$ , the line through  $b'$  and  $b$  has a slope equal to  $h/l$  and is therefore parallel to the line joining the two points of support of the cord. Hence if  $H$  is changed without changing any other conditions of the problem, the pole of the string polygon will move along a line parallel to the line joining the points of support.

### PROBLEMS

214. Four weights of 100 lbs. each are to be suspended by a cord attached to two points of support distant apart 80 ft. horizontally and 20 ft. vertically, the lines in which the weights hang being distant 16 ft., 32 ft., 48 ft., 64 ft. respectively, from one point of support. Construct the string polygons for  $H = 200$  lbs., 400 lbs., 600 lbs., and scale the maximum tension found in the cord in each case. At the same horizontal distance from one support compare the depths of the string polygon, measured from the line joining the points of support of the cord, for different values of  $H$ . Note that these depths vary inversely as the corresponding values of  $H$ .

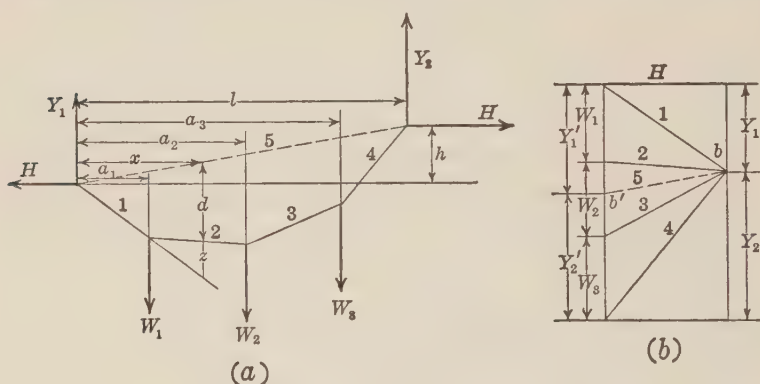


FIG. 188

**87. Depth of the String Polygon.** In Fig. 188 (a) and 188 (b) are shown again the string polygon and the corresponding force polygon. The vertical distance from any point in the string polygon to the line joining the two points of support of the cord is called the *depth* of the string polygon at that point. Thus,  $d$  is the depth of the string polygon at the distance  $x$  from the left

support. Comparing the similar triangles in (a) and (b), we evidently may write the following proportions:  $(d + z)/x = Y_1'/H$ , and  $z/(x - a_1) = W_1/H$ . Eliminating  $z$ , we find

$$d = \frac{xY_1' - (x - a_1)W_1}{H}.$$

Thus we see that if the weights and the lines in which they act are kept unchanged, the depth of the string polygon at any point varies inversely as the horizontal component of the tension in the cord. (The proof can be given for any other point in the same way.)

It follows from this theorem that a string polygon for a set of weights acting in assigned lines can be constructed to have any desired depth at a given point. For a value of  $H$  can be assumed at will and the corresponding string polygon be drawn. If it is desired that the depth of the string polygon be  $d_1$  at a given point and the depth of the polygon constructed is  $d$  at that point, a new value  $H_1$  may be so chosen that  $H_1/H = d/d_1$ , and with this value of  $H_1$  a second string polygon may be constructed which will have the desired depth at the assumed point.

### PROBLEMS

215. Four lamps, each weighing 5 lbs., are to be suspended from a cord supported at points on the same level 100 ft. apart. If the lamps are to be equally spaced horizontally between the supports and the middle lamps are to be 20 ft. below the points of support, find the horizontal tension in the cord, the maximum tension and the lengths of the segments of the cord. Solve graphically as explained above. Also solve algebraically.

(SUGGESTION. Think of the middle segment of the cord as cut and the corresponding tension in the cord put in to hold the portions of the cord in equilibrium. Then consider the portion to one side of the cut as in equilibrium in addition to the equilibrium of the whole cord. Two equations containing the components of the reaction at one support may then be written and the solution completed.)

216. Show that when any set of weights are suspended from a cord supported at two points, the tension in one of the end segments of the cord is greater than half of the sum of the suspended weights.

217. Show how to obtain the ratios of a set of weights to cause a cord to hang in a given string polygon.

218. A cord supported at the ends on the same level carries a load uniformly distributed along the horizontal. Given that the tension in the cord at its

lowest point is one half the total load carried, draw approximately the shape assumed by the cord.

(SUGGESTION. Divide the horizontal distance up into small intervals and assume that the weight in each interval acts at the center of that interval.)

**88. The Linkage Arch.** If, instead of a cord, links capable of taking compression are used, the equilibrium polygon may be put into the form of an arch, as in Fig. 189. The proof and method of construction are similar to those of the cord carrying weights, the

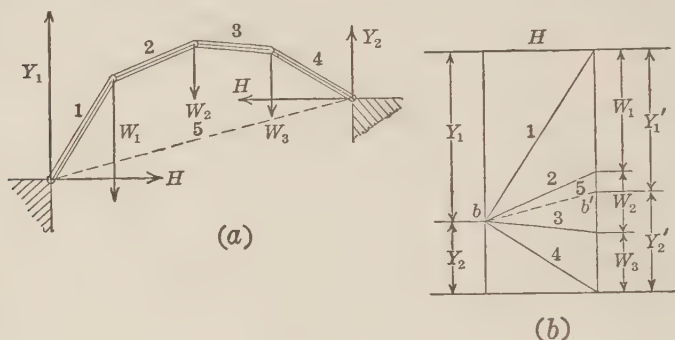


FIG. 189

difference being that the horizontal components of the reactions of the supports act in the opposite directions from those of the cord. There is, however, an important difference in the stability of the two structures. It can be shown that if the linkage arch is disturbed from its position of equilibrium the arch will collapse, while the string polygon, if disturbed, will return to its position of equilibrium.

### PROBLEMS

219. Weights of 20 lbs., 50 lbs., and 40 lbs. are to be supported by a linkage arch in lines distant 3 ft., 5 ft., and 8 ft. respectively from the left support. The right support is distant 10 ft. horizontally and is 4 ft. above the left support. By graphical construction, find the lengths of the links and the forces in them when the horizontal component of the force in each link is 30 lbs. Construct also on the same figure the string polygon for the same weights acting in the same lines and with the same horizontal component of force and find the lengths of the segments of the cord and the tensions in the segments.

220. Ten equal weights are to be supported by a linkage arch. The supports are to be on the same level and the weights are to act in lines equally spaced between the supports. If the end links are inclined  $30^\circ$  to the hori-

zontal, construct the shape of the linkage. Compare with the shape of a string carrying the same weights in the same lines with the end segments inclined  $30^\circ$  with the horizontal.

**89. The Masonry Arch.** When two blocks of stone with plane surfaces are pressed together it is assumed that the variation in pressure follows a straight line law as indicated in Fig. 190, where two blocks, *A* and *B*, are pressed together by forces *P* and *P'*,

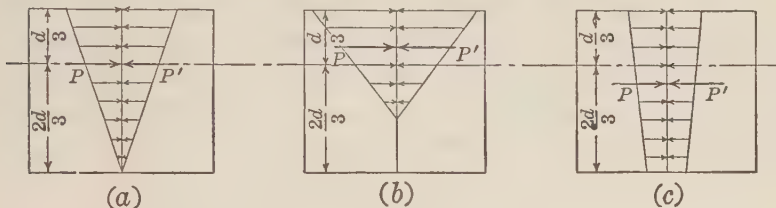


FIG. 190

the depth of the surface in contact being *d* and the numerical values of *P* and *P'* being equal. If the blocks are pressed together in such a way that the pressure at one edge is zero and uniformly increases from that edge to a maximum at the opposite edge, Fig. 190 (a), the resultant force *P* acts at  $2d/3$  from the edge of zero pressure. (The distribution of force is the same as in the weight of a triangular thin slab. Compare Problem 109.) If, as in case (b), the resultant force *P* acts at less than one-third of the depth of the adjacent surfaces from one edge, there will be no pressure at the opposite edge and there will be a tendency for the surfaces to separate there. When, as in case (c), there is pressure at both edges the resultant pressure will fall somewhere within the middle third of the surfaces in contact. Hence in designing an arch it is desirable to keep the *line of thrust* within the middle third of the surfaces of contact, although this is not strictly necessary for equilibrium.

### PROBLEMS

221. Determine approximately, by graphical construction, the form of the line of thrust of an arch to support a load uniformly distributed horizontally over a space of 20 ft., the ends of the arch being at the same level and the top-most part, or *crown*, being 8 ft. above the supports. If the load is 2000 lbs. per horizontal ft., what is the maximum force of compression in the arch? What is the horizontal force exerted by each support?



(SUGGESTION. To construct the form of the arch take the loads as acting at the centers of 1 ft. intervals.)

Also compute accurately the thrust at the crown and at each support by considering the arch as cut at the crown and putting in the forces acting on the two halves of the arch.

222. Solve the above problem if the load starts at 2000 lbs. per ft. at each support and decreases uniformly to 1000 lbs. per ft. at the center, all other conditions remaining unchanged.

**90. Flexible Cord Carrying a Weight Uniformly Distributed along the Horizontal.** (a) *A Numerical Example.* Before discussing the general case, a numerical example will be considered. Let a flexible cord suspended between two points distant 120 ft. apart horizontally and 20 ft. vertically, carry a uniformly dis-

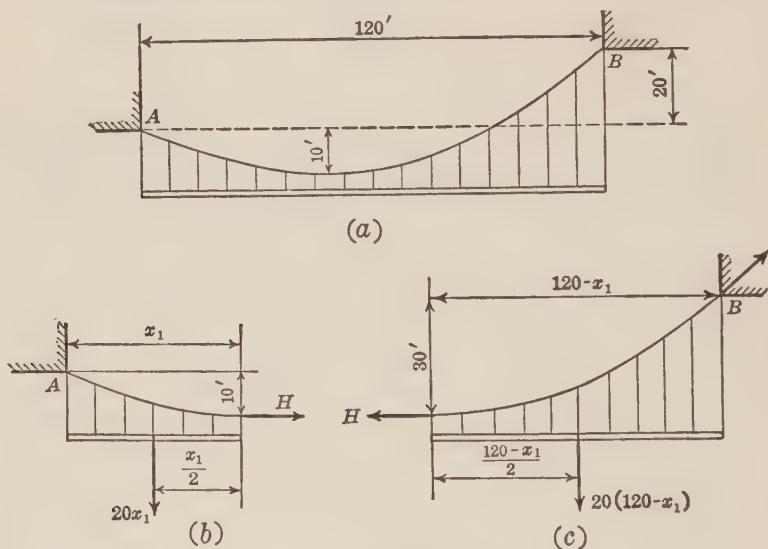


FIG. 191

tributed load of 20 lbs. per horizontal ft., the load being attached to the cord in such a way that any portion of the cord carries the part of the load directly beneath it. If the cord is of such length that the lowest point is 10 ft. below the lower support, find the horizontal distance from the lower support to the lowest point of the cord, the tension in the cord at the lowest point, and the maximum tension (Fig. 191 a).



SOLUTION. Let  $x_1$  be the horizontal distance from the lower support to the lowest point of the cord and  $H$  the tension in the cord at its lowest point. Think of the cord as cut at the lowest point and the horizontal force,  $H$ , acting in the cord there put in on each end of the cut cord. The two portions of the cord may then be thought of as separate bodies in equilibrium under the forces acting on them, as in Fig. 191 (b) and Fig. 191 (c). Since the weight is uniformly distributed along the horizontal, the resultant loads on the two portions of the cord bisect the horizontal projections of these portions, as shown in the figures. Taking moments about  $A$  and  $B$  of the forces holding the two portions of the cord in equilibrium, we get

$$\begin{aligned} 10H &= 20x_1(x_1/2), \\ 30H &= 20(120 - x_1)(120 - x_1)/2. \end{aligned}$$

By division, we find

$$\frac{1}{3} = \frac{x_1^2}{(120 - x_1)^2}.$$

Therefore

$$\frac{1}{\sqrt{3}} = \frac{x_1}{120 - x_1},$$

from which we get

$$x_1 = 43.92 \text{ ft.}, \quad \text{and} \quad H = x_1^2 = 1929 \text{ lbs.}$$

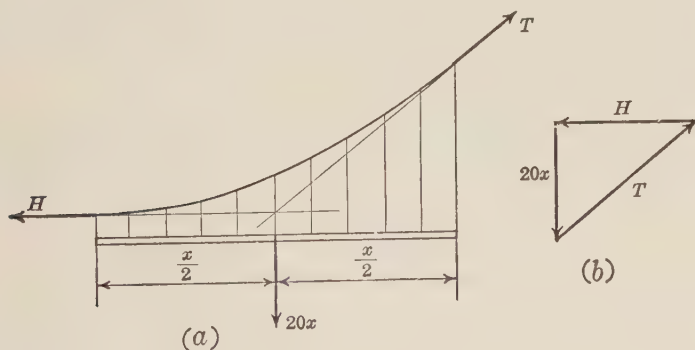


FIG. 192

To find the tension at any other point of the cord, consider a portion of the cord extending from the lowest point to the chosen point, and the forces acting on this portion (Fig. 192 a).

If  $x$  is the horizontal projection of this portion of the cord, and  $T$  is the tension at the chosen point, the forces  $H$ ,  $T$ , and  $20x$  must be in equilibrium and must therefore form a closed (right) triangle, as in Fig. 192 (b). Hence

$$T = (H^2 + (20x)^2)^{1/2}.$$

From this formula it is evident that the maximum value of  $T$  occurs where  $x$  has its maximum value, that is, at the support farthest from the lowest point of the cord. In this case the maximum tension is at the right support where  $x = 76.08$  ft., and its value is

$$\text{Max. } T = \{(1929)^2 + (1521.6)^2\}^{1/2} = 2457 \text{ lbs.}$$

### PROBLEMS

223. Show that if the lowest point of the cord in the example just discussed is taken as the origin of coordinates, with the  $x$  axis horizontal, and if any other point of the cord referred to these axes is  $(x, y)$ , then  $Hy = 10x^2$ , and hence that the cord hangs in the form of a portion of a parabola with axis vertical.

224. A flexible cable carrying a uniformly distributed load of 50 lbs. per horizontal ft. is suspended between two points distant apart 200 ft. horizontally and 50 ft. vertically. If the cord at its lowest point is 40 ft. lower than the lower support, find the position of the lowest point, the horizontal tension, the tension at each support, and the equation of the curve, taking the lowest point as origin and the tangent there as the  $x$  axis.

225. A flexible wire weighing 0.1 lb. per ft. is stretched between two points on the same level 100 ft. apart. If the maximum sag of the wire is 2 ft., find approximately the horizontal tension and the tension at the supports. (Since the wire is so nearly horizontal, it may be considered as carrying a load uniformly distributed along the horizontal.)

226. If a flexible cord carrying a load  $w$  per unit horizontal distance is suspended from two points at the same level distant  $l$  apart and has a deflection  $d$  at the center below the points of support, show that the tension at the lowest point is

$$H = \frac{wl^2}{8d},$$

and that the maximum tension is

$$\text{max. } T = \frac{wl}{8d} (l^2 + 16d^2)^{1/2}.$$

Show also that the equation of the curve in which the cord hangs is

$$ly = 4dx^2,$$

where the origin is at the lowest point of the cord and the tangent at that point is the  $x$  axis.

**90 (b). Discussion of the General Case.** In Fig. 193 (a) is represented a flexible cord supported at  $A$  and  $B$  and carrying a uniformly distributed load  $w$  per horizontal unit distance and so attached to the cord that each portion of the load is carried by the part of the cord directly above it. In Fig. 193 (b) is represented a

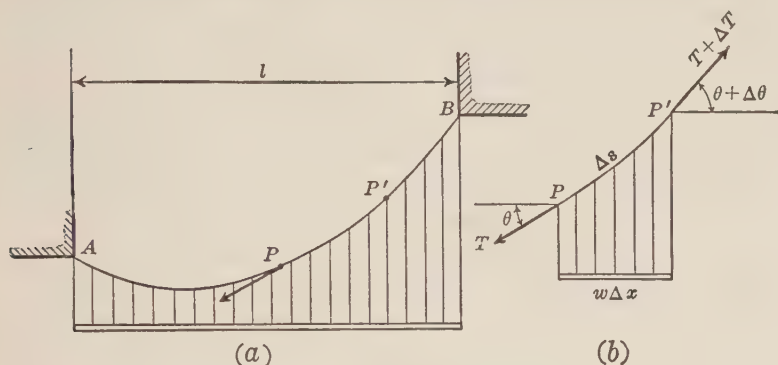


FIG. 193

portion of the cord of length  $\Delta s$  between the points  $P(x, y)$  and  $P'(x + \Delta x, y + \Delta y)$  and the part of the load suspended from that portion. If  $T$  and  $T'$  are the tensions in the cord at  $P$  and  $P'$  and  $\theta$  and  $\theta + \Delta\theta$  the angles which the tensions make with the horizontal, we have for the equilibrium of the portion of the cord, regarded as a rigid body, with the tensions at the ends along the tangents,

$$\begin{aligned}(T + \Delta T) \cos (\theta + \Delta \theta) - T \cos \theta &= 0, \\(T + \Delta T) \sin (\theta + \Delta \theta) - T \sin \theta &= w \Delta x.\end{aligned}$$

By dividing by  $\Delta x$  and passing to the limit, these equations may be reduced to the form

$$\frac{d(T \cos \theta)}{dx} = 0, \quad \frac{d(T \sin \theta)}{dx} = w,$$

from which, by integration, we obtain

$$T \cos \theta = H \text{ (a constant),} \quad T \sin \theta = wx + C.$$

By division we get, since  $\tan \theta = dy/dx$ ,

$$dy = \frac{wx + C}{H} dx.$$

Therefore

$$y = \frac{wx^2}{2H} + \frac{Cx}{H} + C_1.$$

This is the equation of a curve which exists for all real values of  $x$  and which is satisfied by all points on the cord. The origin of coordinates has not been specified in deriving this equation, the sole restriction being that the  $x$  axis is horizontal and the  $y$  axis vertical. Since the equation is of the form  $y = ax^2 + bx + c$ , it represents a parabola with axis vertical. The form of the equation may be simplified by choosing the origin at the lowest point of the curve. This point is not necessarily on the cord, for the cord may hang so that no part of it lies below the lower support, as in Fig. 194 (b). With this choice of the origin, both  $y$  and  $dy/dx$  vanish when  $x = 0$  and the equation of the curve is reduced to

$$y = \frac{wx^2}{2H}.$$

Since  $T \cos \theta = H$  at any point of the cord, the quantity  $H$  is equal to the horizontal component of the tension at any point of

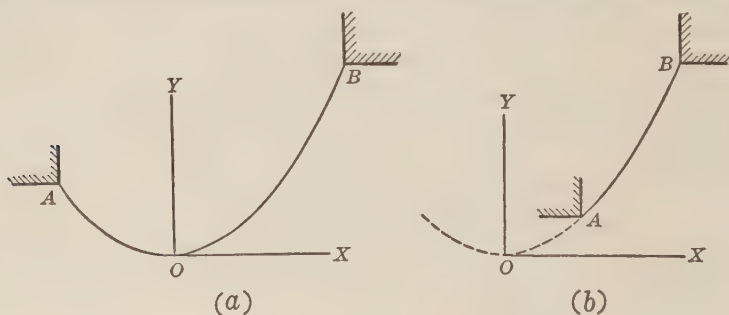


FIG. 194

the cord. If, as in Fig. 194 (a), the lowest point of the cord is below the lower support,  $H$  is equal to the tension at the lowest point of the cord.

The tension  $T$  at any other point  $(x, y)$  on the cord is then given by  $T = H/\cos \theta$ . Since  $\tan \theta = dy/dx = wx/H$ , it follows that

$$\cos \theta = \frac{H}{(H^2 + w^2x^2)^{1/2}},$$

and hence that

$$T = (H^2 + w^2x^2)^{1/2}.$$

From this equation it is evident that the greatest tension in the cord occurs where  $x$  has its greatest value.

If the coordinates of the left and right supports are  $(x_1, y_1)$  and  $(x_2, y_2)$ , respectively, the length of the cord between the supports is

$$s = \int_{x_1}^{x_2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx.$$

If we let  $a = H/w$ , we have  $dy/dx = x/a$ , and hence

$$s = \frac{1}{a} \int_{x_1}^{x_2} (a^2 + x^2)^{1/2} dx,$$

or

$$s = \frac{1}{2a} \left[ x_2(a^2 + x_2^2)^{1/2} - x_1(a^2 + x_1^2)^{1/2} + a^2 \log \frac{x_2 + (a^2 + x_2^2)^{1/2}}{x_1 + (a^2 + x_1^2)^{1/2}} \right].$$

### PROBLEMS

227. A flexible cable is suspended between two points distant apart 250 ft. horizontally and 30 ft. vertically and the lowest point of the cable is 10 ft. lower than the lower point of support. The cable carries a uniformly distributed load of 100 lbs. per horizontal ft. Find the horizontal distance from the lower support to the lowest point of the cable, the horizontal component of the tension in the cable, the maximum tension, and the length of the cable.

228. A cable is to be suspended from two points on the same level distant 120 ft. apart and is to carry a load of 500 lbs. per horizontal ft. If the maximum tension in the cable is to be 50,000 lbs., how much should the cable sag and what length of cable is required (when stretched)?

229. Show that when a cord carrying a uniformly distributed horizontal load is suspended between two points on the same level distant  $l$  apart and the deflection in the center is  $d$ , the length of the cord is

$$s = \frac{1}{2} (l^2 + 16d^2)^{1/2} + \frac{l^2}{8d} \log \frac{4d + (l^2 + 16d^2)^{1/2}}{l}.$$

Use this formula to find the value of  $s$  when  $l = 100$  ft. and  $d = 10$  ft. What increase in  $s$  is necessary to increase  $d$  by one ft.?

**91. Stretching of the Cable.** The length of the cable is increased by the tension. An approximate formula for the amount of elongation can be obtained on the supposition that the material of which the cable is made obeys the law, known as

Hooke's Law, that the elongation of a bar under tension is directly proportional to the applied force. Since the tension varies from point to point in the cable, we must first consider the elongation in an infinitesimal length of the cable. According to Hooke's Law,

$$\text{unit stress/unit strain} = E \text{ (constant).}$$

By *unit stress* at any section of the cable is meant the total tension in the cable at that section divided by the cross-sectional area, and by *unit strain* at any section is meant the elongation of the specimen (assumed to be of constant cross-section and under constant tension throughout its length) divided by its original length. The quantity  $E$  is called the *modulus of elasticity*. In pound and inch units the value of  $E$  for steel ranges from about 28,000,000 to 30,000,000. If stress is reckoned in lbs. per sq. in., since strain is an abstract number, the modulus of elasticity is in lbs. per sq. in.

Let us consider a portion of the wire of original length  $ds$  and cross-sectional area  $A$ . If it is increased under the tension  $T$  by an amount  $de$ , then

$$\frac{T/A}{de/ds} = E,$$

or

$$de = \frac{T}{AE} ds.$$

We have found in § 90 (b) that at any point  $(x, y)$  of the cable the value of the tension is

$$T = (H^2 + w^2x^2)^{1/2}.$$

Moreover,  $T \cos \theta = H$ ; and, since  $\cos \theta = dx/ds$ ,  $ds = Tdx/H$ . Hence

$$de = \frac{T^2 dx}{AEH} = \frac{H^2 + w^2x^2}{AEH} dx.$$

It follows that the elongation of the portion of the cable between points on it whose abscissas are  $x_1$  and  $x_2$ , is

$$e = \frac{1}{AEH} \int_{x_1}^{x_2} (H^2 + w^2x^2) dx.$$



In this discussion it has been assumed that the cable was of constant cross-section throughout and any change in area due to stretching has been neglected.

### PROBLEMS

230. In case the cable carrying a uniformly distributed load along the horizontal is supported at two points on the same level distant  $l$  apart and the deflection is  $d$ , show that the elongation is  $e = wl(3l^2 + 16d^2)/24AE d$ .

231. Using the formula of the preceding problem, assume  $l = 80$  ft.,  $d = 10$  ft.,  $w = 5$  lb./ft., and  $E = 30,000,000$  lbs./sq. in. Choose  $A$ , the cross-sectional area of the cable, so that the maximum tensile stress will be 20,000 lbs./sq. in. and find how much the cable will be stretched.

232. What length of unstretched cable would be required in the preceding problem?

233. What length of unstretched steel cable would be needed to hang between two points on the same level 20 ft. apart if the maximum deflection is to be three inches and  $E = 30,000,000$  lbs./sq. in.?

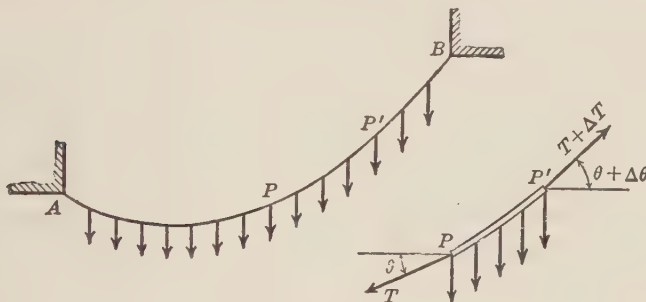


FIG. 195

**92. Flexible Cord Carrying a Load Uniformly Distributed along the Cord.** Let the load carried by the cord be  $w$  per unit length of the cord. Consider a portion of the cord of length  $\Delta s$  between the points  $P$  and  $P'$  (Fig. 195). The forces acting on this element are  $T$ ,  $T + \Delta T$ , and  $w\Delta s$ . Applying the conditions of equilibrium, we have

$$\begin{aligned}(T + \Delta T) \cos (\theta + \Delta \theta) - T \cos \theta &= 0, \\(T + \Delta T) \sin (\theta + \Delta \theta) - T \sin \theta &= w \Delta s.\end{aligned}$$

From these equations we obtain

$$\begin{aligned}d(T \cos \theta) &= 0, \\d(T \sin \theta) &= w ds.\end{aligned}$$

By integration of the first of these equations, we get

$$T \cos \theta = H \text{ (constant).}$$

Substituting this value of  $T$  in the second equation, we get  $d(H \tan \theta) = wds$ , or, since  $\tan \theta = dy/dx$ ,

$$(1) \quad d\left(\frac{dy}{dx}\right) = \frac{w}{H} ds.$$

This is the differential equation of the curve in which the cord hangs. At every point of the cord this equation is satisfied. It does not follow, conversely, that every point on the curve which satisfies this equation is necessarily on the cord. The curve exists for values of  $x$  and  $y$  beyond the extremities of the cord.

Integrating equation (1), we obtain

$$\frac{dy}{dx} = \frac{w}{H} s + C,$$

where  $s$  is the length of the curve measured from some point on the curve. Since this equation is of first degree in  $s$  and  $dy/dx$ , values of  $s$  must exist for all real values of  $dy/dx$ , assuming that  $s$  may be regarded as positive or negative. We may then measure  $s$  from a point on the curve where  $dy/dx$  is zero. If we do so, we have  $s = 0$  when  $dy/dx = 0$ , and hence  $C = 0$ . The differential equation is then reduced to

$$(2) \quad \frac{dy}{dx} = \frac{s}{a}$$

where  $a = H/w$ . Replacing  $dy$  by  $(ds^2 - dx^2)^{1/2}$ , squaring both sides of equation (2) and solving for  $dx$ , we get

$$dx = \frac{ads}{(a^2 + s^2)^{1/2}}.$$

Integrating, we find

$$x = a \log [s + (a^2 + s^2)^{1/2}] + k.$$

If now the origin be chosen on the curve at the point where  $dy/dx = 0$ , we have  $x = 0$  when  $s = 0$ , and hence  $k = -a \log a$ . Therefore

$$x = a \log \frac{s^2 + (a^2 + s^2)^{1/2}}{a}.$$

The exponential form of this equation is

$$s + (a^2 + s^2)^{1/2} = ae^{x/a}.$$

Transposing  $s$ , squaring and solving for  $s$ , we obtain

$$(3) \quad s = \frac{a}{2}(e^{x/a} - e^{-x/a}).$$

This equation expresses  $s$  in terms of  $x$ . To get an equation connecting  $x$  and  $y$ , substitute this value of  $s$  in equation (2). We then have

$$dy = \frac{1}{2}(e^{x/a} - e^{-x/a})dx,$$

from which, by integration, with the condition that  $y = 0$  when  $x = 0$ , we obtain

$$(4) \quad y + a = \frac{a}{2}(e^{x/a} + e^{-x/a}).$$

The curve is called the *catenary*.

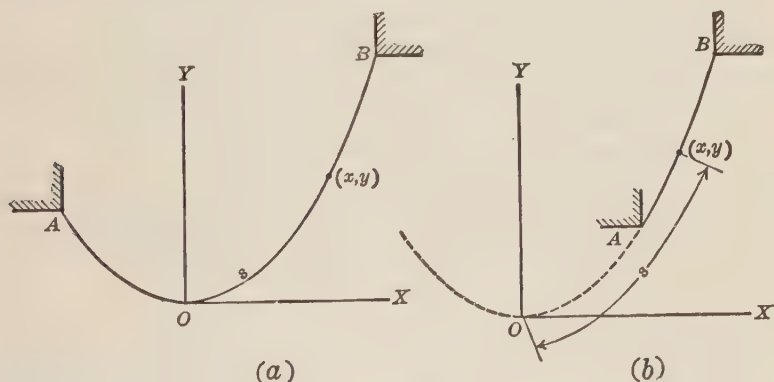


FIG. 196

In Fig. 196 are shown the positions of the origin in the two possible cases of suspension. It must be remembered that  $s$  is the length of arc measured from the origin to  $(x, y)$ .

**93. The Equation of the Catenary in Terms of Hyperbolic Functions.** From the definitions of *hyperbolic sine* and *hyperbolic cosine*:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x}),$$

the equation of the catenary and the formula for the length of arc may be written in the form

$$\begin{aligned}y + a &= a \cosh (x/a), \\s &= a \sinh (x/a).\end{aligned}$$

By using tables of the hyperbolic functions, the labor of computation in the solution of problems involving the catenary is considerably lessened.

#### 94. Some Formulas Connecting Hyperbolic Functions. From

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

the following formulas can be derived:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1, \\ \sinh (-x) &= -\sinh x, \\ \cosh (-x) &= \cosh x, \\ \sinh (x + y) &= \sinh x \cosh y + \cosh x \sinh y, \\ \sinh (x - y) &= \sinh x \cosh y - \cosh x \sinh y, \\ \cosh (x + y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \cosh (x - y) &= \cosh x \cosh y - \sinh x \sinh y, \\ \sinh x + \sinh y &= 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y), \\ \sinh x - \sinh y &= 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y), \\ \cosh x + \cosh y &= 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y), \\ \cosh x - \cosh y &= 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y), \\ d(\sinh x) &= \cosh x \, dx, \\ d(\cosh x) &= \sinh x \, dx.\end{aligned}$$

The truth of the first three of these equations is evident by direct substitution. To derive the fourth formula notice that

$$\begin{aligned}\cosh x + \sinh x &= e^x, \\ \cosh x - \sinh x &= e^{-x},\end{aligned}$$

and hence that

$$\begin{aligned}\sinh (x + y) &= \frac{1}{2}(e^{x+y} - e^{-x-y}) = \frac{1}{2}(e^x e^y - e^{-x} e^{-y}) \\ &= \frac{1}{2}[(\cosh x + \sinh x)(\cosh y + \sinh y) \\ &\quad - (\cosh x - \sinh x)(\cosh y - \sinh y)] \\ &= \sinh x \cosh y + \cosh x \sinh y.\end{aligned}$$

Adding the fourth and fifth equations, we find

$$\sinh (x+y) + \sinh (x-y) = 2 \sinh x \cosh y.$$

If we replace  $x+y$  by  $a$  and  $x-y$  by  $b$  in this equation, we have

$$\sinh a + \sinh b = 2 \sinh \frac{1}{2}(a+b) \cosh \frac{1}{2}(a-b),$$

which is of the same form as the eighth equation of the above set.

By defining  $\tanh x = \sinh x / \cosh x$ ,  $\operatorname{sech} x = 1 / \cosh x$ , etc., additional formulas may be derived from the above set.

### PROBLEMS

234. Derive the above formulas for  $\cosh (x+y)$  and  $\cosh x - \cosh y$ .

235. An inextensible cord weighing  $w$  lbs. per ft. is suspended between two points on the same level 120 ft. apart. What length should the cord have if the horizontal component of the tension is  $50w$ ? What then is the maximum deflection? The maximum tension? Use the tables of hyperbolic functions.

236. Prove that the relation between  $s$  and  $y$  at any point of the catenary is  $(y+a)^2 = s^2 + a^2$ .

237. Show, by using the formula of the preceding problem, that when a cord of length  $c$  and weight  $w$  per unit length is suspended between two points at the same level and sags a distance  $d$  below the line joining these points, the value of the horizontal component of the tension is

$$H = \frac{c^2 - 4d^2}{8d} w.$$

238. Prove that the tension  $T$  in the cord carrying a load uniformly distributed along the cord, the weight per unit length being  $w$ , at a distance  $s$  measured along the curve from the lowest point of the curve is

$$T = (H^2 + w^2 s^2)^{1/2} = w(a+y) = H \cosh (x/a).$$

239. At what distance apart on the same level should the ends of a cord of length 50 ft. be supported so that the maximum tension in the cord shall be 100 times the weight of one foot of the cord?

240. A flexible cord carrying a weight of  $w$  lbs. per foot of length is suspended between two points distant apart 200 ft. horizontally and 40 ft. vertically. If the horizontal component of the tension is  $1000w$ , find the deflection of the lowest point of the cord below the lower point of support, the length of the cord, and the maximum tension in the cord.

SOLUTION. Assume the lowest point of the curve to divide the horizontal distance between the points of support into  $x_1$  and  $x_2$ , and let its distance below the lower support be  $d$  (Fig. 197). Then  $x_1 + x_2 = 200$  and the points of support are  $A(-x_1, d)$  and  $B(x_2, d+40)$ . Also  $a = H/w = 1000$ . From the equation of the curve,

$$\begin{aligned} d+40+1000 &= 1000 \cosh (x_2/1000), \\ d+1000 &= 1000 \cosh (x_1/1000), \end{aligned}$$

since

$$\cosh(-x) = \cosh x.$$

By subtraction,

$$\begin{aligned} 40 &= 1000 [\cosh(x_2/1000) - \cosh(x_1/1000)] \\ &= 2000 \sinh\left(\frac{1}{2} \frac{x_2 + x_1}{1000}\right) \sinh\left(\frac{1}{2} \frac{x_2 - x_1}{1000}\right) \\ &= 2000 \sinh 0.1 \sinh\left(\frac{x_2 - x_1}{2000}\right). \end{aligned}$$

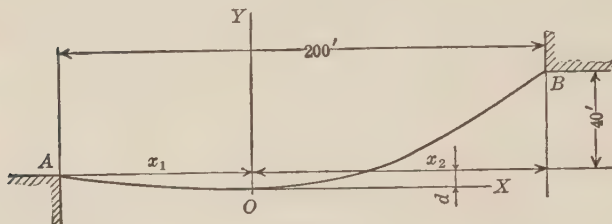


FIG. 197

Therefore

$$\sinh \frac{x_2 - x_1}{2000} = \frac{1}{50 \sinh 0.1}.$$

From the table of hyperbolic functions,  $\sinh 0.1 = 0.1002$  and hence

$$\sinh\left(\frac{x_2 - x_1}{2000}\right) = 0.1996,$$

from which

$$\frac{x_2 - x_1}{2000} = 0.1983,$$

and

$$x_2 - x_1 = 396.6.$$

Combining this with  $x_2 + x_1 = 200$ , we get

$$x_2 = 298.3 \text{ ft.}, \quad x_1 = -98.3 \text{ ft.}$$

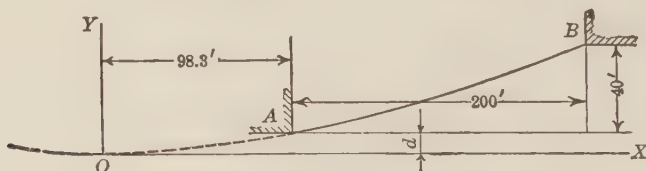


FIG. 198

The result shows that the cord does not hang below the lower support, but that the lowest point of the curve is 98.3 ft. beyond the lower support, as in Fig. 198. The value of  $d$  is found from

$$d + 1000 = 1000 \cosh\left(\frac{98.3}{1000}\right),$$



or

$$\begin{aligned} d &= 1000(\cosh 0.0983 - 1), \\ &= 1000(1.0048 - 1) = 4.8 \text{ ft.} \end{aligned}$$

The length of cord from  $A$  to  $B$  is

$$\begin{aligned} s &= 1000 \left( \sinh \frac{298.3}{1000} - \sinh \frac{98.3}{1000} \right), \\ &= 1000(0.3027 - 0.0985), \\ &= 204.2 \text{ ft.} \end{aligned}$$

It is interesting to compare this value of  $s$  with the straight line distance from  $A$  to  $B$  which is

$$(200^2 + 40^2)^{1/2} = 203.96 \text{ ft.}$$

The maximum tension is

$$T = w(a + y) = w(1000 + 44.8) = 1045 w.$$

241. Solve the above problem if  $H = 100 w$ .

242. A wire is suspended between two points 500 ft. apart on the same level. If the deflection at the center is 20 ft. and the wire weighs 0.1 lb. per ft., what is the tension at the middle, the maximum tension, and the length of the wire?

SOLUTION. The coordinates of one support are (250, 20). Therefore

$$20 + a = a \cosh \frac{250}{a}.$$

This equation can be solved only by trial. Writing it in the form

$$\frac{20}{a} + 1 = \cosh \frac{250}{a},$$

we may make the following table.

| $a$ (trial values) | $1 + \frac{20}{a}$ | $\cosh \frac{250}{a}$ | Difference |
|--------------------|--------------------|-----------------------|------------|
| 200 .....          | 1.1                | 1.89                  | - 0.79     |
| 400 .....          | 1.05               | 1.20                  | - 0.15     |
| 1000 .....         | 1.02               | 1.0314                | - 0.011    |
| 2000 .....         | 1.01               | 1.0078                | 0.0022     |
| 1800 .....         | 1.0111             | 1.0097                | 0.0014     |
| 1500 .....         | 1.0133             | 1.0139                | - 0.0006   |
| 1600 .....         | 1.0125             | 1.0122                | 0.0003     |
| 1570 .....         | 1.01274            | 1.0127                | 0.00004    |
| 1560 .....         | 1.01282            | 1.0128                | 0.00002    |

Hence the value 1560 for  $a$  may be accepted as correct to the degree of accuracy attainable with the five place tables. Then

$$H = wa = 156 \text{ lbs.}$$

The length of the wire is therefore

$$\begin{aligned} s &= 2a \sinh (250/a) = 3120 \sinh (0.16026), \\ &= 3120(0.1610) = 502.3 \text{ ft.} \end{aligned}$$

The maximum tension is

$$T = w(a + y) = 0.1(1560 + 20) = 158 \text{ lbs.}$$

243. Assuming the wire in the preceding problem to be of steel weighing 490 lbs. per cu. ft., show that the maximum tensile stress in the wire is about 5380 lbs. per sq. in. Show also that if the modulus of elasticity is 30,000,000 lbs./sq. in., the wire is stretched about 1.08 in.

244. A steel wire of diameter  $1/8$  in., weighing 490 lbs./cu. ft., is suspended from two points 200 ft. apart on the same level. If the central deflection is 20 ft., find the maximum tension, the length of wire, and the approximate amount of elongation if  $E = 30,000,000$  lbs./sq. in.

245. A wire of length 60 ft., when stretched between two points on the same level, sags 3 ft. below the points of support. Find the value of  $H/w$  and the distance between the supports.

246. A cord of length 102 ft. is suspended between two points 100 ft. apart on the same level. Find the deflection at the center, assuming the cord to be inextensible.

**95. Elongation of a Wire Suspended between Two Points and Carrying a Load Uniformly Distributed along the Wire.** In § 92 we found that  $T \cos \theta = H$  and  $dy/dx = ws/H = s/a$ . Now  $\cos \theta = dx/ds$  and hence

$$T = H \frac{ds}{dx} = H \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} = H \left( 1 + \frac{s^2}{a^2} \right)^{1/2}.$$

If now we consider a portion of the wire of unstretched length  $ds$

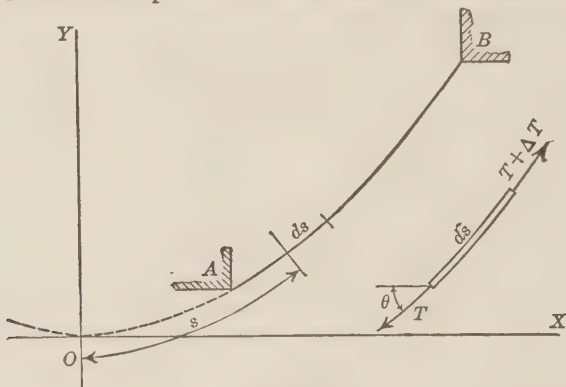


FIG. 199

and assume that the elongation of this portion is proportional to the tension in it, we have for the elongation  $de$  of this portion,

$$de = \frac{T ds}{AE}$$

where  $A$  is the cross-sectional area of the wire, assumed constant, and  $E$  is the modulus of elasticity of the material (Fig. 199). Substituting the above value of  $T$ , replacing  $s/a$  by  $\sinh (x/a)$  and  $ds$  by  $Tdx/H$ , we obtain,

$$\begin{aligned} de &= \frac{T^2 dx}{HAE} = \frac{H}{AE} \left( 1 + \sinh^2 \frac{x}{a} \right) dx \\ &= \frac{H}{AE} \cosh^2 \frac{x}{a} = \frac{H}{2AE} \left( 1 + \cosh \frac{2x}{a} \right) dx. \end{aligned}$$

Therefore the elongation of any portion of the cord between two points whose abscissas are  $x_1$  and  $x_2$  is

$$e = \frac{H}{2AE} \int_{x_1}^{x_2} \left( 1 + \cosh \frac{2x}{a} \right) dx.$$

#### PROBLEMS

247. Prove that the elongation of a wire suspended between two points on the same level distant  $l$  inches apart and carrying  $w$  lbs. per inch of wire is

$$e = \frac{H}{2AE} \left( l + a \sinh \frac{l}{a} \right) \text{ inches,}$$

where  $A$  is the cross-sectional area of the wire in sq. in.,  $E$  is the modulus of elasticity in lbs./sq. in.,  $H$  = the horizontal component of the tension in lbs., and  $a = H/w$ .

248. Find the elongation of the wire of Problem 243 by the formula of the preceding problem and compare the result with that found in Problem 243. (Neglect the change in weight of a unit length of wire and the change in cross-sectional area due to the stretching of the wire.)

## CHAPTER VI

### DYNAMICS OF A PARTICLE; ROTARY MOTION OF A BODY

**96. Displacement, Velocity, and Acceleration in Straight-Line Motion.** Consider a particle, or a point, in motion. Its position is determined at any instant by its distance and direction from a fixed point. The directed distance of a particle from a fixed point is called the *displacement* of the particle with respect to the fixed point. From the definition it is evident that displacement is a vector quantity.

The analysis of the motion of a particle moving along a straight line, in general, is simpler than that of motion in a curve, and for that reason certain types of straight-line motion will be considered first.

In straight-line motion, if the point of reference is taken on the line of motion, the vector nature of displacement loses significance. It is sufficient in this case to distinguish distances on one side of the reference point from those on the other side, and this can be done by the use of plus and minus signs. Thus, choosing the point  $O$ , Fig. 200, as a point of reference, or origin, a particle

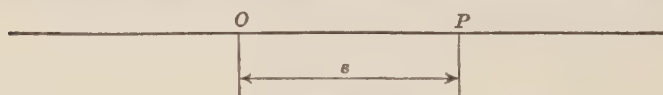


FIG. 200

$P$  on the line  $OX$  is said to have a positive displacement when it is to the right of  $O$ , and a negative displacement when it is to the left of  $O$ . The displacement may then be represented by the value of  $s$ , where  $s = OP$ , and is to be regarded as positive or negative according as  $P$  lies to the right or left of  $O$ .

If a particle moves in a straight line, its displacement in any interval of time divided by the time, is called the *average velocity* of the particle during that interval. The limiting value of the

average velocity of the particle for an interval of time, as the length of the interval of time approaches zero, is called the *velocity* of the particle at the beginning of the interval, or the *velocity at the instant*. Thus, in Fig. 200, if a particle is at  $P$  with displacement  $s$ , at the time  $t$ , and at  $P'$  with displacement  $s + \Delta s$  at the time  $t + \Delta t$ , the average velocity of the particle during the time  $\Delta t$  is  $\Delta s/\Delta t$ , and the value of the velocity at the time  $t$  is  $ds/dt$ . Since  $\Delta s$ , or  $PP'$ , is a directed quantity, and  $\Delta t$  is a scalar quantity, the ratio  $\Delta s/\Delta t$  is a directed quantity and has the same direction as  $\Delta s$ , or from  $P$  to  $P'$ . Then  $ds/dt$  is a directed quantity, in general, and has the direction assumed by  $PP'$  as  $P'$  approaches  $P$  as a limit. Hence if  $ds/dt$  is positive, the particle is moving in the direction in which  $s$  increases (to the right in Fig. 200), and if  $ds/dt$  is negative the particle is moving in the direction in which  $s$  decreases.

If a particle moves in a straight line, its change in velocity during an interval of time, divided by the time, is called the *average acceleration* of the particle in that interval. The limiting value of the average acceleration, as the interval of time approaches zero, is called the *acceleration at the instant*. Thus if  $v$  is the velocity of the particle at any time  $t$  and  $v + \Delta v$  is the velocity at the time  $t + \Delta t$ , the value of  $\Delta v/\Delta t$  is the average acceleration of the particle during the time  $\Delta t$  and  $dv/dt$  is the acceleration of the particle at the time  $t$ . Since  $\Delta v$  is a directed quantity along the line of motion,  $dv/dt$  is also a directed quantity along the line of motion, and its positive direction is the same as the positive direction for displacement and velocity.

Since  $v = ds/dt$ , the acceleration,  $dv/dt$ , may be written in the form

$$\frac{d^2s}{dt^2}, \quad \text{or} \quad \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds}.$$

Hence we have, for straight-line motion,

$$\text{displacement} = s,$$

$$\text{velocity} = v = \frac{ds}{dt},$$

$$\text{acceleration} = a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \frac{dv}{ds}.$$

It must not be assumed that these formulas hold for other than straight-line motion.

If  $s$  is measured in feet and  $t$  in seconds, velocity is in feet per second (ft./sec.) and acceleration in feet per second per second (ft./sec.<sup>2</sup>).

**97. Illustration.** Let us suppose that a certain particle moves on a vertical line, and that its displacement  $s$  from a fixed point  $O$ , is expressed in terms of the time  $t$  by the formula

$$s = 10 + 96t - 16t^2,$$

where  $s$  is measured in feet from  $O$  (with positive values of  $s$  for positions above  $O$ ), and  $t$  is measured in seconds from a given instant. To find the displacement, velocity, and acceleration of the particle at any later instant.

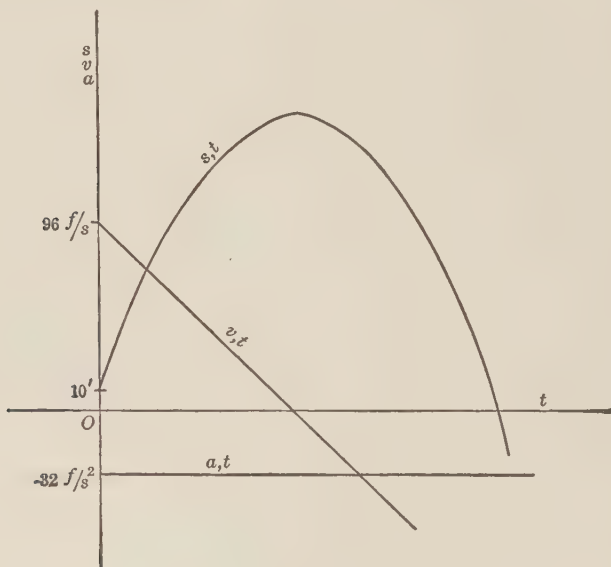


FIG. 201

We have seen that the velocity and acceleration are directed quantities along the line of motion of the particle, and that the magnitudes of these quantities are obtained from the displacement by differentiation. Thus we have

$$v = ds/dt = 96 - 32t \text{ ft./sec.}$$

$$a = dv/dt = -32 \text{ ft./sec}^2.$$



These relations may be shown graphically by plotting curves which represent  $s$ ,  $v$ , and  $a$ , respectively, in terms of  $t$ . Let  $t$  be used as abscissa and  $s$ ,  $v$ , and  $a$  as ordinates, counting to the right as positive for abscissas and upward as positive for ordinates. The curves are then as represented in Fig. 201.

### PROBLEMS

249. In the illustration of § 97, for what values of  $t$  is the displacement positive and increasing, positive and decreasing, negative? At what value of  $t$  does the velocity change from positive to negative?

250. Given that a particle moves along a straight line with a constant acceleration  $-20$  ft./sec.<sup>2</sup>, find the equations which express the velocity in terms of the time, and the displacement in terms of the time, having given that the velocity is 64 ft./sec. when  $t = 0$  and that the displacement is 30 ft. when  $t = 0$ . Sketch the curves which represent the displacement, velocity and acceleration in terms of the time.

251. A particle moves along a straight line so that the displacement is given by the equation  $s = 2 \cos 3t$ , in which  $t$  is in seconds, and  $s$  in feet. Derive the expression for the velocity in terms of the time and the acceleration in terms of the time. Express also the acceleration in terms of the displacement. Sketch the curves which show the displacement in terms of the time, the velocity in terms of the time, the acceleration in terms of the time, and the acceleration in terms of the displacement.

252. Given that a particle moves along a straight line with an acceleration  $a = -12 \cos 2t$  ft./sec.<sup>2</sup>, find the values of the velocity in terms of the time and the displacement in terms of the time, under the conditions that the velocity is zero when the time is zero and that the displacement is 3 ft. when the time is zero. Sketch the curves of acceleration-time, velocity-time, and displacement-time.

253. A particle moves along a straight line toward a fixed point  $O$  on the line; when the displacement of the particle from  $O$  is  $s$  ft., the acceleration is  $-10/s^2$  ft./sec.<sup>2</sup>. Find the value of the velocity of the particle when at a displacement  $s$  from  $O$ , having given that the particle starts from rest at 1000 ft. from  $O$ .

(SUGGESTION. Write the acceleration in the form  $v dv/ds$ , equate to  $-10/s^2$ , separate the variables and integrate, determining the constant of integration by the condition that  $v = 0$  when  $s = 1000$  ft.)

254. A small body in space, under the attraction of the earth only, would move from rest toward the earth with an acceleration varying inversely as the square of the distance from the earth's center. Having given that the acceleration at the earth's surface is 32.2 ft./sec.<sup>2</sup>, and that the earth's radius is 3956 miles, show that at any distance,  $s$ , of the body from the center of the earth, the value of the acceleration is

$$v \frac{dv}{ds} = - \frac{32.2(3956)^2(5280)^2}{s^2}$$

where  $s$  is measured in feet. Then show that if the body starts from rest at an

infinite distance from the earth, it would reach the earth's surface with a velocity of about 6.95 mi./sec., neglecting the resistance of the earth's atmosphere.

255. A particle is projected from a point  $O$  with a velocity  $v_0$  and has an acceleration directed toward  $O$  which varies as its distance from  $O$ . Show that its equation of motion is  $v dv/ds = -ks$ , where  $k$  is a positive constant quantity, and then show that the velocity of the particle when at a distance  $s$  from  $O$  is  $v = \pm (v_0^2 - ks^2)^{1/2}$ . How far from  $O$  will the particle go? Describe the subsequent motion.

256. A particle is projected vertically upward with an initial velocity  $v_0$  and has a constant downward acceleration  $g$  (32.2 ft./sec<sup>2</sup>). If  $s$  is the displacement upward from the point of projection, show, by taking the form  $v dv/ds$  for  $a$ , that at any height  $s$  above the point of projection, the velocity is given by  $v^2 = v_0^2 - 2gs$ . To what height will the particle rise?

**98. Newton's Laws of Motion.** Sir Isaac Newton in 1686, as the result of close observation of natural phenomena, formulated the following three laws:

1. *Every body continues in its state of rest or uniform motion in a straight line unless impelled by external force to change that state.*
2. *Rate of change of momentum is proportional to the force acting and takes place in the direction in which the force acts.*
3. *To every action there is an equal and opposite reaction.*

In interpreting the above laws a "body" should be considered to be a "material particle," that is, a body so small that its mass may be considered to be concentrated at a point. For the "mass" of a body we shall accept the familiar, though rather unsatisfactory, definition of "quantity of matter" in the body, a quantity which may be determined by comparison with standard quantities, pound, gram, etc., by use of the balance.

"Momentum," used in the second law, means the product of mass and velocity, and "rate of change of momentum" is equivalent to product of mass and acceleration. Thus, if  $m$  is the mass of the particle,  $a$  its acceleration, and  $F$  the force acting upon it, the second law may be formulated as  $F = kma$  where  $k$  is a constant. The value of  $k$  depends upon the units of force, mass, and acceleration used. Suppose a mass of  $W$  lbs. is used and let the force acting upon it be measured in lbs. and the acceleration in ft./sec<sup>2</sup>. A body having a mass of  $W$  lbs. is attracted by the earth at the earth's surface by a force of  $W$  lbs. Let this mass of

$W$  lbs. be released under the earth's attraction. Experiment has shown that it has, under the action of the earth's attraction an acceleration of  $g$  ft./sec.<sup>2</sup>, where  $g$  is approximately equal to 32.2. The equation  $F = kma$  then becomes  $W = kWg$ , or  $k = 1/g$ . Then for any force, using the force in pounds, the mass in pounds, and the acceleration in feet-per-second squared, the equation may be written  $F = (W/g)a$ . This equation is often written in the form

$$F = ma,$$

where  $m = W/g$ , and is called the mass. In this case the unit of mass is equivalent to  $g$  pounds. The term *slug* has been proposed for this unit of mass and has had some acceptance. Hereafter in this book, when the unit of mass is not specified, it will be understood to be this unit.

The above equation applies to bodies of any size provided the motion is one of translation only; that is, where all particles of the body have the same motion at any instant.

**99. Illustration.** A car weighing two tons is equipped with a coiled spring which requires a force of 10 000 lbs. to compress it one inch. If the car is moving with a velocity of 5 ft./sec. when the spring comes in contact with a bumping post (Fig. 202), how much will the spring be compressed, assuming that the post remains stationary and that the force required to compress the spring through any distance varies directly as the amount it is compressed?

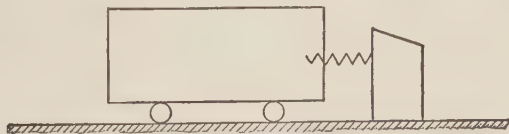


FIG. 202

Let the direction in which the car is moving be considered positive. Then if  $s$  is the distance in ft. that the car has moved after coming in contact with the post, the acceleration is  $v dv/ds$  and the equation of motion is

$$- 120\,000s = \frac{4000}{32.2} v \frac{dv}{ds}.$$

Multiplying by  $ds$  and integrating, we obtain

$$- 30s^2 = v^2/32.2 + C.$$

When  $s = 0$ ,  $v = 5$ ; therefore  $C = -25/32.2$ , and

$$v^2 = 25 - 966s^2.$$

The car stops when  $966s^2 = 25$ , or  $s = 0.161$  ft. = 1.93 in. The time required for the car to come to rest may be obtained by integrating a second time:

$$\begin{aligned} v &= \frac{ds}{dt} = (25 - 966s^2)^{1/2}, \\ dt &= \frac{ds}{(25 - 966s^2)^{1/2}}, \\ t &= \frac{1}{\sqrt{966}} \sin^{-1} \frac{s\sqrt{966}}{5} + C. \end{aligned}$$

When  $s = 0$ ,  $t = 0$ . Hence  $C = 0$ . When the car stops,  $s = 5/\sqrt{966}$ ,  $t = (\sin^{-1} 1)/\sqrt{966} = 0.051$  sec. Of course the car does not remain in this position, since the spring then exerts against the car a force of (1.93)(10 000) or 19 300 lbs.

### PROBLEMS

257. A car weighing  $W$  lbs. is moving on a smooth level track with a velocity of  $v_0$  ft./sec. It is equipped with a spring which requires  $k$  lbs. for each inch of compression. If it strikes a post which does not yield, show that the car will move an additional distance  $(12Wv_0^2/kg)^{1/2}$  inches, and that the time required is  $\pi\sqrt{W}/\sqrt{48kg}$  sec., where  $g = 32.2$ . (Note that the time is independent of  $v_0$ .)

258. A body of weight  $W$  lbs. is moving along a horizontal plane and is acted upon by a constant frictional force of  $kW$  lbs. If the velocity of the body at a given instant is  $v_0$  ft./sec., show that the body will move an additional distance  $v_0^2/(2kg)$  ft., and also that the time required for it to come to rest is  $v_0/(kg)$  sec.

259. A body of weight  $W$  is projected up a smooth plane inclined at an angle of  $20^\circ$  with the horizontal with a velocity of 80 ft./sec. Show that it will move a distance of 290.6 ft. up the plane and will take 7.26 sec. for the upward motion.

260. Show that when a body is projected with velocity  $v_0$  up a smooth plane that is inclined at an angle  $\theta$  with the horizontal, the distance it will ascend is  $v_0^2/(2g \sin \theta)$  and that the time required for the ascent is  $v_0/(g \sin \theta)$ . Prove also that the body will return in the same time to the starting point and will reach that point with velocity  $v_0$ .

261. A body of weight  $W$  lbs. is projected with velocity  $v_0$  ft./sec. up a plane inclined at an angle  $\theta$  with the horizontal. A constant friction force  $F$  lbs. retards the motion of the body. Prove that it will move up the plane a distance

$$\frac{Wv_0^2}{2g(W \sin \theta + F)} \text{ ft.},$$

and that the time of the ascent is

$$\frac{Wv_0}{g(W \sin \theta + F)} \text{ sec.}$$

Show also that it will return to the starting point with a velocity

$$v_0 \left[ \frac{W \sin \theta - F}{W \sin \theta + F} \right]^{1/2},$$

and that the time required for the descent is

$$\frac{W v_0}{g[(W \sin \theta + F)(W \sin \theta - F)]^{1/2}} \text{ sec.}$$

Compare these results when  $F$  is zero with those of the preceding problem.

262. A body falls under the action of its own weight and the resistance of the air, which at any instant is approximately equal to the product of a constant  $k$  and the square of the velocity of the body at that instant. Show that when it has descended a distance of  $s$  ft. from rest the velocity is given by

$$v^2 = \frac{W}{k} (1 - e^{-2kgs/W}).$$

What value does  $v$  approach as  $s$  increases indefinitely? If  $W = 100k$ , what is this limiting value? Sketch the curve showing  $v$  in terms of  $s$ .

263. A body of weight  $W$  is projected vertically upward with initial velocity  $v_0$ . If it is acted upon by its own weight and a variable resistance  $kv^2$ , show that it will rise to a height

$$h = \frac{W}{2kg} \log \left( 1 + \frac{kv_0^2}{W} \right)$$

and that it will return to the starting point with velocity  $v_0[W/(W + kv_0^2)]^{1/2}$ .

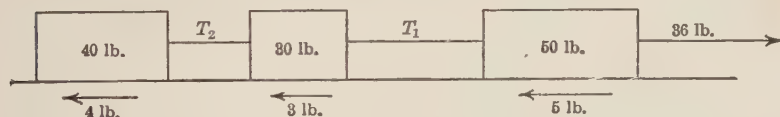


FIG. 203

264. Three weights connected as shown in Fig. 203 are drawn along a horizontal plane by a force of 36 lbs. The frictional forces acting on the bodies are respectively 5 lbs., 3 lbs., and 4 lbs. Prove that the acceleration given to the bodies is 6.44 ft./sec<sup>2</sup>. and that the tensions in the connecting cords are  $T_1 = 21$  lbs.,  $T_2 = 12$  lbs.

(SUGGESTION. Write the equations of motion for the three bodies separately; thus for the first body to the right the equation of motion is

$$36 - 5 - T_1 = \frac{50}{g} a.$$

If the equations for the motion of the other two bodies are written, the tensions may be eliminated by the addition of the equations and a single equation be obtained containing the one unknown,  $a$ .)

265. In Fig. 204 the 50-lb. weight is acted upon by a friction force of 4 lbs. What weight  $W$  is required to move the system 10 ft. from rest in 2 sec.?

266. The horizontal planes in Fig. 205 are smooth. The cord connecting the weights  $A$  and  $B$  slips freely through a ring attached to  $C$ . If the system



is released from rest, how far will each weight move in one second, and what will be the tension in the cord? (Note that the distance moved by  $C$  is one-half the sum of the distances moved by  $A$  and  $B$ , and that therefore a corresponding relation exists between the accelerations of the three weights.)

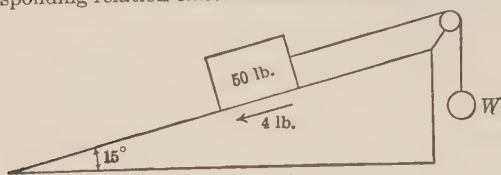


FIG. 204

267. A particle moves with constant acceleration  $a$ . If the initial velocity is  $v_0$  and  $v$  is the velocity  $t$  seconds later, show that the following formulas hold:

$$v = v_0 + at, \quad s = \frac{1}{2}at^2 + v_0t, \quad v^2 - v_0^2 = 2as.$$

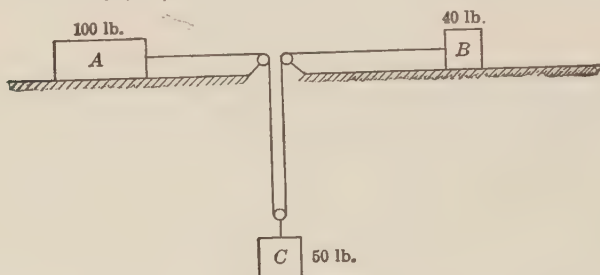


FIG. 205

268. In Fig. 206 a cord is attached to  $A$  and  $B$  and runs freely through a ring attached to  $C$ . The planes are smooth. The weights  $A$ ,  $B$ , and  $C$  are respectively 50 lbs., 40 lbs., and 60 lbs. If  $s_1$ ,  $s_2$ , and  $s_3$  are the respective distances moved from rest by  $A$ ,  $B$ , and  $C$  in  $t$  seconds,  $a_1$ ,  $a_2$ , and  $a_3$  the corresponding accelerations, and  $T$  the tension in the cord, show that  $2s_3 = s_1 + s_2$  and that hence  $2a_3 = a_1 + a_2$ . Write the equation of motion for each weight and by combining them show that

$$\begin{aligned} a_1 &= 21.87 \text{ ft./sec.}^2, \\ a_2 &= 23.31 \text{ ft./sec.}^2, \\ a_3 &= 22.59 \text{ ft./sec.}^2, T = 8.955 \text{ lbs.} \end{aligned}$$

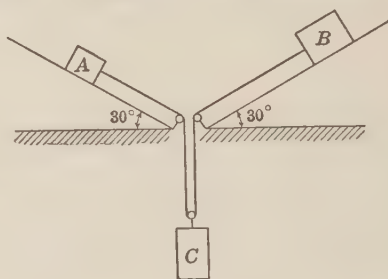


FIG. 206

269. Two equal weights  $W$  are drawn along smooth horizontal planes by a cord attached to them and passing through a smooth ring attached to a weight  $W_1$  as shown in Fig. 207. Show that the weights  $W$  will have an acceleration  $\frac{1}{2}$  that of a freely falling body according as  $W_1 \geq 4W$ .

270. Weights  $W$  and  $W_1$  are connected as shown in Fig. 208. Show that the acceleration of the weight  $W$  up the smooth plane is

$$a = \frac{(4W_1 - W)g}{8W_1 + 2W}$$

and hence that  $W$  will move up or down according as  $4W_1 \geq W$ .



271. Two equal weights,  $A$  and  $B$ , move along smooth planes as indicated in Fig. 209. The cord attached to  $B$  and  $C$  slips through a smooth ring attached to  $A$ . If  $a_1$ ,  $a_2$ , and  $a_3$  are respectively the accelerations of  $A$ ,  $B$ , and  $C$ ,  $a_1$  and  $a_2$  being up the planes and  $a_3$  down, show that

$$a_1 = \frac{(5W_1 - W)g}{2(5W_1 + W)}, \quad a_2 = -\frac{Wg}{2(5W_1 + W)}, \quad a_3 = \frac{(10W_1 - 3W)g}{2(5W_1 + W)},$$

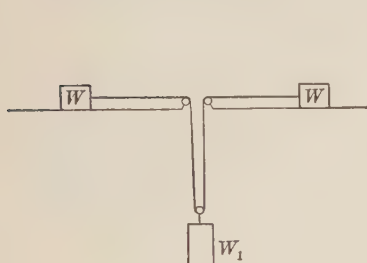


FIG. 207

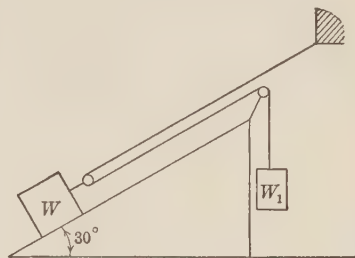


FIG. 208

where  $W$  is the weight of  $A$  and of  $B$  and  $W_1$  is the weight of  $C$ . (Note that if the system is released from rest,  $B$  will move down the plane, that  $A$  will move up or down according as  $5W_1 \geq W$ , and that  $C$  will move down or up according as  $10W_1 \geq 3W$ .) What ratio of  $W_1$  to  $W$  will hold  $A$  stationary?  $C$  stationary? For what range of  $W_1 : W$  will  $A$  and  $C$  both move up?

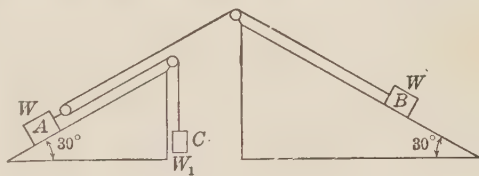


FIG. 209

272. A body of weight  $W$ , having an initial velocity  $v_0$ , moves along a smooth horizontal plane under the action of a resisting force equal to  $kWv^n$ , where  $v$  is the velocity at any instant. Show that the value of the space,  $s$ , that the body will move is finite only when  $n < 2$ . Show also that in this case the limiting distance is  $v_0^{2-n}/(2-n)kg$  and that it is attained in a finite time only when  $n < 1$ .

(SUGGESTION. Use  $v dv/ds$  to get space and  $dv/dt$  to get time.)

273. A man and parachute weigh 150 lbs. The resistance of the air to the open parachute varies as the square of the velocity and as the projected area on a plane perpendicular to the line of motion. If this area is 100 sq. ft., and if the resistance is 1 lb. per sq. ft. of area when the velocity is 20 ft./sec., at what velocity will motion be uniform? If a velocity of 60 ft./sec. is attained before the parachute is opened, how far will the man descend before the velocity is reduced to 30 ft./sec.? What will the velocity be after a descent of 10 ft. after the parachute opens? After a descent of 40 ft.? If the weight of the man is 140 lbs., what will be the greatest tension in the rope supporting him?

274. A toboggan with its load weighs 400 lbs. It slides down an incline of 1 ft. vertical to 20 ft. of track. The friction is 4 lbs. and the air resistance is  $v^2/100$  lbs., where  $v$  is the velocity in ft./sec. Starting from rest, what is the velocity after moving 50 ft.? What limiting value will the velocity approach on a long slide?

**100. Simple Harmonic Motion.** A particle is said to have *simple harmonic motion* when it moves along a straight line and has at each instant an acceleration directed toward a fixed point of the line and proportional to its distance from that point.



FIG. 210

We shall now derive from this definition equations connecting displacement and velocity, velocity and time, and displacement and time.

Let  $P$  be a particle which moves along the line  $OX$  (Fig. 210), having at each instant an acceleration directed toward  $O$  and proportional to the distance of  $P$  from  $O$ . Let  $s = OP$  be the displacement of  $P$  from  $O$ , regarded as positive when  $P$  is to the right of  $O$ , and as negative when  $P$  is to the left of  $O$ . Then the acceleration of  $P$  is in any position equal to  $-ks$ , where  $k$  is a positive quantity. For if  $P$  is to the right of  $O$ ,  $s$  is positive and  $-ks$  is therefore directed toward the left; while if  $P$  is to the left of  $O$ ,  $s$  is negative and  $-ks$  is therefore directed toward the right; in either case the acceleration is toward  $O$ . We have then for any position of the particle,

$$v \frac{dv}{ds} = -ks.$$

Multiplying by  $ds$  and integrating, we obtain

$$v^2 = -ks^2 + C.$$

As  $s$  increases  $v$  decreases and becomes zero for a certain value of  $s$ . Let  $s = s_1$  when  $v = 0$ ; then  $C = ks_1^2$  and we have

$$v^2 = k(s_1^2 - s^2).$$

This equation shows that the velocity reduces to zero when  $s$  becomes either  $s_1$  or  $-s_1$ , and that the numerical value of the velocity depends only upon the numerical value of  $s$ , irrespective

of whether the particle is on the right or left of  $O$ , and irrespective of whether it is moving to the right or to the left. The particle therefore oscillates periodically between the points represented by  $s_1$  and  $-s_1$ .

From the above equation we have

$$v = \pm \sqrt{k} \sqrt{s_1^2 - s^2}.$$

Since  $s$  increases toward the right,  $ds/dt$  represents the velocity toward the right and  $-ds/dt$  the velocity toward the left. Hence while the particle is moving toward the right,

$$\frac{ds}{dt} = \sqrt{k} \sqrt{s_1^2 - s^2},$$

or

$$dt = \frac{1}{\sqrt{k}} \frac{ds}{\sqrt{s_1^2 - s^2}},$$

from which

$$t = \frac{1}{\sqrt{k}} \sin^{-1} \frac{s}{s_1} + C.$$

Let  $t = 0$  when  $s = 0$ ; then  $C = - (1/\sqrt{k}) \sin^{-1} 0 = 0$ , say. As  $t$  increases  $\sin^{-1} (s/s_1)$  must increase, and hence when  $s$  becomes  $s_1$ ,  $\sin^{-1} (s/s_1)$  becomes  $\pi/2$  and  $t = \pi/(2\sqrt{k})$ . This value of  $t$  is therefore one-fourth of the time it takes the particle to make one complete vibration. Hence the *period* of a complete vibration is

$$T = \frac{2\pi}{\sqrt{k}}.$$

The equation

$$t = \frac{1}{\sqrt{k}} \sin^{-1} \frac{s}{s_1}$$

may be written in the form

$$s = s_1 \sin (\sqrt{k}t),$$

from which we obtain by differentiating

$$\frac{ds}{dt} = s_1 \sqrt{k} \cos (\sqrt{k}t).$$

This equation gives the value of the velocity in terms of the time measured from the instant when the particle is at the center of its motion and is moving to the right.

## PROBLEMS

275. A particle oscillates with simple harmonic motion between two points 16 inches apart. The period of a complete vibration is 0.5 sec. Find the maximum velocity. Find the displacement and velocity 0.4 sec. after passing through the center in the positive direction. What is the velocity when the particle is 4 inches from the center?

276. A weight of 50 lbs. is attached to one end of a spring and can move along a smooth horizontal plane as indicated in Fig. 211. The other end of

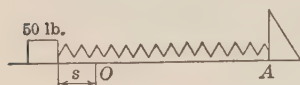


FIG. 211

the spring is attached to a firm support at A. It requires a force of 20 lbs. to compress or stretch the spring 1 inch, and the amount of compression or elongation is proportional to the force applied. If the movable end of the spring is at O when the spring is at its

natural length, show that if the weight is pulled aside and released when the spring is stretched 3 inches, the motion of the weight will be determined by the equation

$$-240s = \frac{50}{g} v \frac{dv}{ds},$$

where  $s$  is the displacement in feet of the weight from its position of equilibrium. Show that the motion is simple harmonic with O as center; and by integration of the above equation, find the period of vibration and the maximum velocity.

277. A point Q moves with constant speed around the circumference of a circle of radius  $r$ , turning through  $\omega$  radians per second. If P is the projection of Q upon a diameter (Fig. 212), show that the displacement,  $s$ , of P from the center of the circle is  $s = r \sin \omega t$ , that the velocity,  $v$ , is  $v = r\omega \cos \omega t$ , and the acceleration,  $a$ , is  $a = -\omega^2 r \sin \omega t = -\omega^2 s$ . Prove therefore that P moves with simple harmonic motion and that the period is  $2\pi/\omega$  sec.

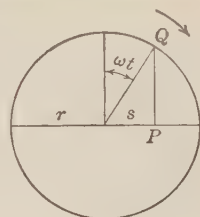


FIG. 212

278. A spiral spring hangs vertically, the upper end being fixed (Fig. 213). A weight W attached to the lower end of the spring stretches it  $s_0$  ft. when hanging at rest. Show that if the weight is pulled down an additional distance  $s_1$  ft. and released it will oscillate with simple harmonic motion in a period  $2\pi\sqrt{s_0/g}$  sec. through the interval  $2s_1$ .



FIG. 213

(SUGGESTION. After the weight has been pulled down and released, if  $s$  is the additional extension of the spring beyond  $s_0$  at any time, the tension,  $T$ , in the spring at that instant is given by  $T/W = (s_0 + s)/s_0$ , and the equation of motion of the weight is

$$T - W = -\frac{W}{g} v \frac{dv}{ds},$$

which, by the substitution for  $T$ , is reduced to

$$-\frac{g}{s_0} s = v \frac{dv}{ds} = a.$$

Compare this with the general equation of simple harmonic motion of § 100. By the comparison show that different weights attached to the same spring will have periods of vibration proportional to the square roots of the weights.)

279. When a thin bar is placed in a horizontal position and its ends firmly fixed (Fig. 214), the deflection caused by a force at the center varies directly as the force. Show that if a weight attached to



FIG. 214

the middle of the bar will deflect it  $s_0$  ft. when at rest, it will vibrate with simple harmonic motion if pulled down an additional distance and released, and the period of vibration will be  $2\pi\sqrt{s_0/g}$  sec.

280. A spring is composed of portions of different springs (Fig. 215), each of which is uniform and obeys Hooke's law, *i.e.* the extension varies directly as the force applied. Show that the whole spring obeys a law of the same form, and that if a suspended weight deflects the spring a distance of  $s_0$  ft. when hanging at rest, it will vibrate with simple harmonic motion if pulled down an additional distance and released, and the period of vibration will be  $2\pi\sqrt{s_0/g}$  sec.



FIG. 215

(It should be noted that in the above problems the effect of the mass of the spring on the motion has been neglected. The solution is therefore a close approximation only in case the mass of the spring is small in comparison with the mass of the attached weight. A like remark applies to other problems of this set.)

281. Given that the deflection of a bar when rigidly supported at the ends is proportional to the force applied at the middle, and the deflection of a coiled spring is proportional to the force applied, show that the total deflection when the bar rests on two equal springs at the ends, as in Fig. 216, is proportional to the force applied at the middle. Then show that when a weight is attached to the center of the bar and displaced vertically from the position of equilibrium



FIG. 216

and released, it will vibrate with simple harmonic motion with a period  $2\pi\sqrt{s_0/g}$  sec., where  $s_0$  is the deflection in ft. with the weight at rest.



FIG. 217

282. A weight suspended from one spring vibrates with a period  $T_1$ , and when suspended from a second spring vibrates with a period  $T_2$ . Show that if the springs are attached end to end, as in Fig. 217, and the same weight is attached, the period of vibration will be  $(T_1^2 + T_2^2)^{1/2}$ . Extend the case to include three or more springs.

**101. Velocity and Acceleration of a Particle in Plane Curvilinear Motion.** When a particle moves along a curved path from  $A$  to  $B$ , the *average velocity* of the particle between  $A$  and  $B$  is defined to be the velocity that the particle would have if it moved uniformly from  $A$  to  $B$ , that is, along the chord  $AB$ ,



traversing equal distances in equal times. Thus (Fig. 218) if the chord  $AB = \Delta c$  and the time required for the particle to move from  $A$  to  $B$  is  $\Delta t$ , the average velocity between  $A$  and  $B$  is  $\Delta c/\Delta t$

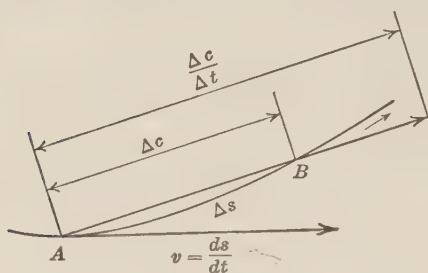


FIG. 218

and its direction is along  $AB$  from  $A$  towards  $B$ . The limiting value of this ratio as  $\Delta t$  approaches zero is defined to be the **velocity** at  $A$ . Its value is therefore  $dc/dt$  and its direction is that of the tangent to the curve at  $A$  and in the direction of the motion.

If the arc  $AB = \Delta s$ , it is shown in calculus that the limit of  $\Delta c/\Delta s$  is 1, or  $dc = ds$ . Hence the velocity,  $v$ , is equal to  $ds/dt$  and is represented by a vector of that value laid off from the point  $A$  along the tangent and in the direction of the motion of the particle.

The vector gain in velocity of the particle in going from a position  $A$  to a position  $B$  on a curve, divided by the time taken between  $A$  and  $B$ , is called the **average acceleration** of the particle in that motion, and the limiting value of this average acceleration as  $B$  is moved along the curve to coincide with  $A$  is called the **acceleration** of the particle at the point  $A$ . Thus, in Fig. 219, if  $v$  and  $v + \Delta v$  are the numerical values of the velocities at  $A$  and  $B$ , and are represented by vectors drawn tangent to the curve at these points, their vector difference,  $PQ$ , divided by the time,  $\Delta t$ , taken by the particle in going from  $A$  to  $B$ , is the average acceleration of the particle between  $A$  and  $B$ . This average acceleration is a vector quantity and has the direction of

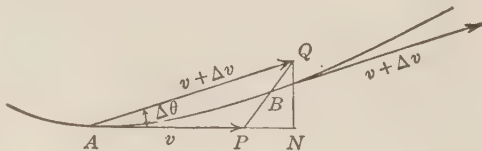


FIG. 219

$PQ$ . Before we pass to the limit, it is more convenient to resolve the average acceleration into two components, one along the tangent to the curve, the other normal to the curve, toward



the concave side. In Fig. 219 these components are equal to  $PN$  and  $NQ$ , respectively. If  $a_t$  and  $a_n$  respectively, are the tangential and normal components of the acceleration of the particle when at  $A$ , then

$$a_t = \lim_{\Delta t \rightarrow 0} \frac{PN}{\Delta t}, \quad \text{and} \quad a_n = \lim_{\Delta t \rightarrow 0} \frac{NQ}{\Delta t}.$$

If  $\Delta\theta$  is the angle between the tangents at  $A$  and  $B$ ,

$$PN = (v + \Delta v) \cos \Delta\theta - v, \quad NQ = (v + \Delta v) \sin \Delta\theta.$$

Therefore

$$\begin{aligned} a_t &= \lim_{\Delta t \rightarrow 0} \frac{-v(1 - \cos \Delta\theta) + \Delta v \cos \Delta\theta}{\Delta t} \\ &= -\lim_{\Delta t \rightarrow 0} \frac{v 2 \sin^2 (\Delta\theta/2)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} \cos \Delta\theta \\ &= -v \lim_{\Delta t \rightarrow 0} \frac{\sin (\Delta\theta/2)}{(\Delta\theta/2)} \frac{\Delta\theta}{\Delta t} \sin (\Delta\theta/2) + \frac{dv}{dt} \\ &= -v \cdot 1 \cdot \frac{d\theta}{dt} \cdot 0 + \frac{dv}{dt} \\ &= \frac{dv}{dt}. \\ a_n &= \lim_{\Delta t \rightarrow 0} \frac{(v + \Delta v) \sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta s} \frac{\Delta s}{\Delta t} \\ &= v \cdot 1 \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt}. \end{aligned}$$

But  $ds/dt = v$  and  $ds/d\theta = \rho$ , the radius of curvature of the curve at  $A$ , and hence  $a_n = v^2/\rho$ . Hence the tangential and normal components of the acceleration of the particle at any point are, respectively,

$$a_t = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{d^2s}{dt^2}, \quad a_n = \frac{v^2}{\rho}.$$

Since, by Newton's second law, the force acting on a particle is equal to the product of the mass of the particle and its acceleration and acts in the direction of the acceleration, the component forces acting on the particle along the tangent and normal to the

curve are respectively

$$F_t = M \frac{dv}{dt}, \quad F_n = M \frac{v^2}{\rho},$$

where  $M = W/g$ , and where  $F_n$  is directed toward the concave side of the curve.

**102. Illustrations.** **EXAMPLE 1.** A particle of weight  $W$ , attached to one end of a weightless, inextensible string of length  $l$ , is placed on a perfectly smooth horizontal plane table and the other end of the string is attached to a point in the surface of the table. The string is stretched and the particle given a velocity  $v$  perpendicular to the string. The forces acting on the particle after the initial velocity is imparted to it are the weight, the force exerted on the particle by the table, and the tension in the string. Since the table top is smooth, it can exert a force only at right angles to the surface. The string can exert a force only when stretched and then only along the string. There is then no force acting along the tangent to the path, which, by the preceding paragraph, means that there is no tangential acceleration. Hence the value of  $ds/dt$  is constant, or the particle moves with constant speed in its path. The string prevents the particle from moving in a straight line and must furnish the force normal to the path which causes it to deviate from a straight line. Hence the string is kept stretched, and the particle moves in a circular path with constant speed  $v$ . The force along the normal to the path is in this case the tension,  $T$ , in the string, and hence

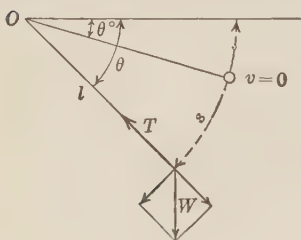


FIG. 220

$$T = \frac{W}{g} \cdot \frac{v^2}{l}.$$

**EXAMPLE 2.** A particle of weight  $W$  is attached by a very light thread to a fixed point  $O$ . The thread is stretched and the weight is released from rest when the thread is at an angle  $\theta_0$  below the horizontal; to find the tension in the thread and the velocity

of the particle when the thread is at an angle  $\theta$  below the horizontal (Fig. 220).

If  $s$  is the arc measured from the horizontal through  $O$ , and  $l$  the length of the thread, then

$$s = l\theta, \quad \frac{ds}{dt} = l \frac{d\theta}{dt}, \quad \frac{d^2s}{dt^2} = l \frac{d^2\theta}{dt^2}.$$

The forces acting on the particle are the weight  $W$  and the tension,  $T$ , in the thread. The tangential component of the resultant force is therefore  $W \cos \theta$ , and the component normal to the path (toward the concave side) is  $T - W \sin \theta$ . Hence, by § 101,

$$(1) \quad W \cos \theta = \frac{W}{g} \cdot l \frac{d^2\theta}{dt^2},$$

$$(2) \quad T - W \sin \theta = \frac{W}{g} \cdot \frac{v^2}{l}.$$

If we set  $d\theta/dt = \omega$ ,  $d^2\theta/dt^2$  may be written

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt} \left( \frac{d\theta}{dt} \right) = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta},$$

and equation (1) is reduced to the form

$$\cos \theta = \frac{l}{g} \omega \frac{d\omega}{d\theta},$$

or

$$g \cos \theta d\theta = l\omega d\omega$$

from which

$$g \sin \theta = \frac{1}{2} l \omega^2 + C.$$

When  $\theta = \theta_0$ ,  $\omega = 0$ , and therefore  $C = g \sin \theta_0$ , and

$$l\omega^2 = 2g(\sin \theta - \sin \theta_0).$$

But  $v = ds/dt = l d\theta/dt = l\omega$ , and therefore

$$\frac{v^2}{l} = 2g(\sin \theta - \sin \theta_0),$$

or

$$v^2 = 2gl(\sin \theta - \sin \theta_0).$$

Substituting this value of  $v$  in equation (2), we obtain

$$T = W(3 \sin \theta - 2 \sin \theta_0).$$

### PROBLEMS

283. Show from the equations derived in the preceding illustration that the particle will rise on the opposite side of the vertical to the height at which it started. Show also that if the weight was released when the string was horizontal, the maximum tension induced in the string is three times the weight of the particle.

284. A small body of weight 5 lbs. slides down a smooth circular track of radius 10 ft. in a vertical plane. When  $30^\circ$  from the lowest point of the path

the velocity is 12 ft./sec. Find the tangential force, the normal force, and the force exerted by the track on the weight in the given position.

285. A one-pound weight attached to one end of a string is made to swing around the other (fixed) end of the string in a vertical circle. When  $45^\circ$  from its highest position the velocity of the weight is 40 ft./sec. If the length of the string is 5 ft., what is the tension in the string in the given position?

286. A boy of weight 100 lbs. swings in a swing in which the supporting ropes are 20 ft. long. If his velocity at the lowest position is 30 ft./sec., what is the tension in each of the two ropes, assuming his weight to be concentrated at the ends of the ropes?

287. A small bead of weight  $W$  is placed on a smooth wire. The wire is bent into the form of a circle of radius  $r$  and placed in a vertical plane, and the bead, starting from rest at the top, slides around the circle. Show that when it has traversed a central angle  $\theta$  the velocity is given by  $v^2 = 2gr(1 - \cos \theta)$  and that the pressure of the wire against the bead (outward) is  $W(3 \cos \theta - 2)$ . Then show that the pressure of the wire against the bead changes from outward to inward when  $\theta$  is about  $48^\circ 11'$ . Show also that when the bead reaches the lowest point of the circle the pressure is  $5W$ .

**103. Acceleration in Terms of Rectangular Coordinates.** If a particle in moving along a curve is at a given instant at a point  $(x, y)$

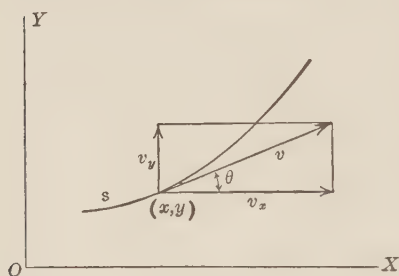


FIG. 221

the velocity, being a vector, may be resolved into components parallel to the coordinate axes as follows: if  $v_x$  and  $v_y$  represent respectively the components of the velocity parallel to the  $x$  and  $y$  axes, and  $\theta$  the angle which the tangent to the curve makes with the  $x$  axis, we have

$$v_x = v \cos \theta, \quad v_y = v \sin \theta$$

(Fig. 221). But  $v = ds/dt$ ,  $\cos \theta = dx/ds$ ,  $\sin \theta = dy/ds$ , and hence

$$v_x = \frac{ds}{dt} \frac{dx}{ds} = \frac{dx}{dt}, \quad v_y = \frac{ds}{dt} \frac{dy}{ds} = \frac{dy}{dt}.$$

The same result is obtained by regarding the component of the particle's velocity parallel to an axis as the velocity of the projection of the particle on that axis. Thus the component of the velocity parallel to the  $x$  axis is the rate at which the abscissa of the particle is increasing, or  $dx/dt$ .

In like manner the components of the acceleration of the particle parallel to the axes may be interpreted as the rates of change of the component velocities, or as the accelerations of the projections of the particle on the axes. Thus, calling the components of the acceleration parallel to the  $x$  and  $y$  axes respectively  $a_x$  and  $a_y$ , we have

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

**104. Illustration.** A small body slides from rest at  $A$  (Fig. 222) along a smooth curved track in a vertical plane, leaving the track at  $O$ , which is 20 ft. vertically below  $A$ , where the tangent to the track makes an angle of  $30^\circ$  with the horizontal. To find the velocity at  $O$ , the horizontal distance from  $O$  at which the body strikes a horizontal plane 15 ft. below  $O$ , and the magnitude and direction of the velocity with which it strikes.

There are two parts of the motion to consider, (a) while the body is on the track, (b) after it leaves the track.

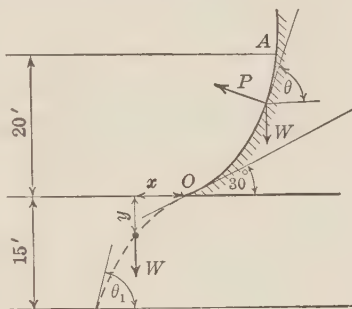


FIG. 222

(a) While on the track the forces acting on the body are  $W$ , the weight, and the force  $P$  exerted by the track, called the “reaction” of the track. Since the track is smooth this force can have no component along the tangent.  $P$  is therefore normal to the track. The tangential force is therefore  $W \sin \theta$ . We have therefore, by § 101,

$$W \sin \theta = \frac{W}{g} \frac{dv}{dt} = \frac{W}{g} \cdot v \cdot \frac{dv}{ds}.$$

But  $\sin \theta = dy/ds$ , where  $y$  is the vertical distance below  $A$  and  $s$  is the length of the path measured from  $A$ . Hence

$$g dy = v dv,$$

from which  $gy = v^2/2 + C$ . When  $y = 0$ ,  $v = 0$ , and hence  $C = 0$ . Therefore  $v^2 = 2gy$ , and at  $O$ ,  $v_0^2 = 40g$ .

(b) After leaving the track the only force acting is the weight. Let us take  $O$  as origin, and let  $x$  be counted positive to the left and  $y$  positive downward. Equating the force parallel to each axis to the

product of mass and acceleration parallel to that axis, we find

$$\frac{d^2x}{dt^2} = 0, \quad \frac{W}{g} \frac{d^2y}{dt^2} = W.$$

Integrating these equations, we have

$$\frac{dx}{dt} = C_1, \quad \frac{dy}{dt} = gt + C_2.$$

When  $t = 0$ ,  $dx/dt = \sqrt{40g} \cos 30^\circ = C_1$ , or  $C_1 = \sqrt{30g}$ ,

$$\frac{dy}{dt} = \sqrt{40g} \sin 30^\circ = C_2, \quad \text{or} \quad C_2 = \sqrt{10g}.$$

Integrating again, we find

$$x = \sqrt{30g} \cdot t + C_3, \quad y = gt^2/2 + \sqrt{10g} \cdot t + C_4.$$

When  $t = 0$ ,  $x = y = 0$ , and hence  $C_3$  and  $C_4$  are both zero. Eliminating  $t$ , we find the equation

$$y = \frac{x^2}{60} + \frac{x}{\sqrt{3}},$$

which shows that after the body leaves the track the path is a parabola. Substituting  $y = 15$  in this equation, the resulting value of  $x$  is  $10\sqrt{3}$ , or 17.32 ft.

The velocity with which the weight strikes the plane is given by finding the value of  $t$  when  $x = 10\sqrt{3}$  and substituting this value in  $dx/dt$  and  $dy/dt$ . The results are

$$t = \sqrt{10/g} \text{ sec.}, \quad dx/dt = \sqrt{30g} \text{ ft./sec.}, \quad dy/dt = 2\sqrt{10g} \text{ ft./sec.}$$

Combining  $dx/dt$  and  $dy/dt$ , we obtain the value of  $v$

$$v = \{(dx/dt)^2 + (dy/dt)^2\}^{1/2} = (30g + 40g)^{1/2} = 47.48 \text{ ft./sec.}$$

The direction of  $v$  is given by

$$\tan \theta_1 = v_y/v_x = \sqrt{40g}/\sqrt{30g} = 2/\sqrt{3},$$

whence

$$\theta_1 = 49^\circ 6'.$$

### PROBLEMS

288. A small sphere of weight 5 lbs. is attached to one end of a rod of length 6 ft., of negligible weight, the other end of which is pivoted on a smooth horizontal axis. If the weight is started with initial velocity 0 when it is directly above the axis, find the velocity of the weight and the force exerted by the rod on the weight when the rod has turned through  $60^\circ$ ,  $90^\circ$ ,  $150^\circ$ ,  $180^\circ$ . At what position will the force in the rod change from compression to tension?



289. An iron sphere of weight 50 lbs. is attached by a rope 10 ft. long, from center of sphere, to a fixed point. The sphere is pulled aside until the rope is horizontal and released. Find, approximately the velocity of the center of the sphere after turning through  $60^\circ$ ,  $90^\circ$ .

290. A small weight starts at  $A$  and slides down the smooth track which ends at  $B$  at an angle of  $30^\circ$  with the horizontal (Fig. 223). Find at what height above  $B$  the weight should start from rest so as to reach  $C$ , 10 ft. below and 20 ft. horizontally from  $B$ .

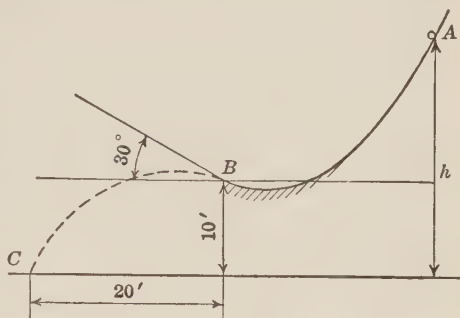


FIG. 223

291. A particle is projected with velocity  $v_0$  ft./sec. at an angle  $\alpha$  with the horizontal (Fig. 224). Take for positive  $x$  the direction of the horizontal component of  $v_0$ , and take  $y$  positive upward. Neglecting air resistance, show that the coordinates  $x$  and  $y$  of the particle after it has been in motion for  $t$  seconds are

$$x = v_0 t \cos \alpha,$$

$$y = v_0 t \sin \alpha - \frac{gt^2}{2}.$$

Eliminate  $t$  between these equations and show that the equation of the path is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha},$$

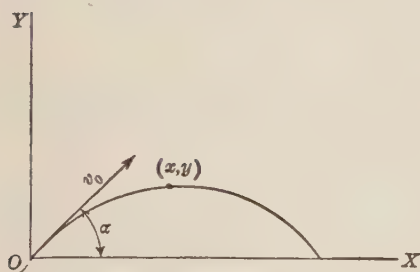


FIG. 224

which is a parabola. Show that the maximum height attained is  $v_0^2 \sin^2 \alpha / (2g)$  ft., and is reached at the end of  $v_0 \sin \alpha / g$  sec., and that the horizontal range is  $v_0^2 \sin 2\alpha / g$  ft. Prove also that the horizontal range is a maximum when  $\alpha = 45^\circ$ , and that its value is  $v_0^2 / g$  ft.

292. A projectile is fired with velocity  $v_0$  ft./sec. at an inclination  $\alpha$  to the horizontal from the foot of a plane inclined  $\beta$  to the horizontal (Fig. 225). Neglecting air resistance, show that the range on the inclined plane is

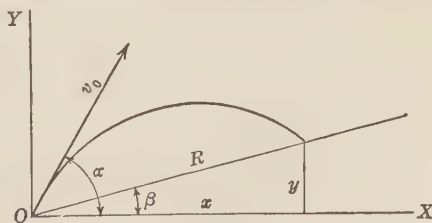


FIG. 225

$$2v_0^2 \cos \alpha \sin (\alpha - \beta) / (g \cos^2 \beta),$$

that the maximum range is equal to  $v_0^2/[g(1 + \sin \beta)]$ , and that this value is attained when  $\alpha = 45^\circ + \beta/2$ . Find also the maximum range down such a plane.

(SUGGESTION. In the equation of the path found in the preceding problem put  $x = R \cos \beta$ ,  $y = R \sin \beta$ , and solve for  $R$ .)

293. A boy throws a stone a distance of 95 yds. on the level. If the angle of elevation is  $35^\circ$ , what initial velocity is required? With the same velocity what is the maximum distance the stone could be thrown?

294. A bullet is fired horizontally with an initial velocity of 2000 ft./sec. Air resistance neglected, how much will the bullet drop in a horizontal distance of 100 ft.? 1000 ft.?

295. For high velocities the resistance of the air modifies considerably the path of a projectile. This resistance is usually considered to vary as some power of the velocity. Assume that in the case of the bullet fired horizontally, the negative acceleration due to the air resistance is  $-kv_x^2$ , since  $v$  and  $v_x$  are nearly the same, and that the  $y$  acceleration is due to gravity only. Show then that, measured from the point of projection,

$$x = (1/k) \log(v_0 kt + 1), \quad y = -gt^2/2,$$

and

$$y = -g(e^{kx} - 1)^2/(2v_0^2 k^2).$$

If, in the case of the bullet of the preceding problem, the value of  $k$  is 0.012, show that there is a drop of 1.81 in. in a distance of 100 ft. from the point of projection.

296. From a cliff a stone is thrown with a velocity of 110 ft./sec. At what angle of elevation should it be thrown so as to reach the greatest horizontal distance on a level plain 200 ft. below the point of projection, and what is the maximum distance?

(SUGGESTION. Take the point of projection as origin and write the equation of the path as found in Problem 291. Substitute  $-200$  for  $y$  and differentiate the resulting equation with respect to  $\alpha$ , putting  $dx/d\alpha = 0$  in the result to obtain the relation between  $\alpha$  and  $x$  when  $x$  is a maximum. Combine this equation with the equation from which it was obtained to find  $\alpha$  and the maximum range.)

297. A projectile is fired from a height  $h$  above a horizontal plane with an initial velocity  $v_0$ . Show that the maximum range on the plane is  $v_0(v_0^2 + 2gh)^{1/2}/g$ , and is attained when the tangent of the angle of elevation is  $v_0/(v_0^2 + 2gh)^{1/2}$ .

298. In Fig. 226 is shown the range,  $r$ , of a projectile fired from  $O$  on a plane inclined at an angle  $\phi$  with the horizontal, the angle between  $r$  and the line of the plane having greatest inclination being  $\theta$ . Show that

the inclination  $\beta$  of  $r$  to the horizontal, satisfies the equation  $\sin \beta = \sin \phi \cos \theta$ .

If then  $r$  is the maximum range that can be attained in that direction, show

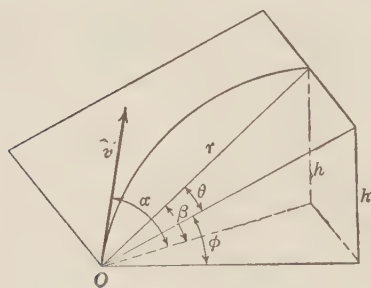


FIG. 226

that

$$r = v_0^2 / [g(1 + \sin \phi \cos \theta)].$$

(Compare Problem 292.) Hence show that as  $\theta$  changes the end of  $r$  traces out an ellipse with  $O$  as focus, the major diameter being on a line of the plane of greatest inclination.

### 105. Motion of a Particle on a Smooth Path in a Vertical Plane.

Let a particle be constrained to move along a smooth path  $AB$  in a vertical plane under the action of its own weight and the reaction of the path (Fig. 227). Since the path is smooth, the reaction,  $F$ ,

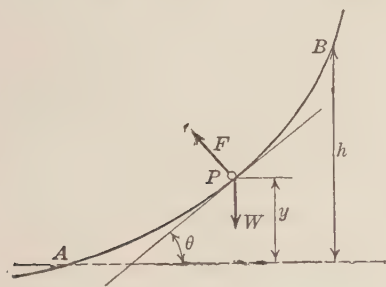


FIG. 227

of the path against the particle must be normal to the path at each point. Let the vertical distance of  $B$  above  $A$  be  $h$  and the vertical distance of  $P$  above  $A$  be  $y$ , where  $P$  is the position of the particle at any instant. Let  $v_a$ ,  $v$ , and  $v_b$  be the velocities of the particle at  $A$ ,  $P$ , and  $B$ , respectively. When the par-

ticle is at  $P$  the component of the resultant force along the tangent is  $W \sin \theta$ , where  $\theta$  is the inclination of the tangent to the path at  $P$ .

(1) If the particle is moving from  $A$  to  $B$ , let  $s = AP$ ; then  $v dv/ds$  is the acceleration along the path from  $P$  toward  $B$ . Since the component of the tangential force is opposite in direction to this acceleration, we have

$$\frac{W}{g} v \frac{dv}{ds} = -W \sin \theta.$$

Here  $\sin \theta = dy/ds$ , and hence

$$\frac{v}{g} \frac{dv}{ds} = -\frac{dy}{ds},$$

from which by integration we obtain

$$\frac{v^2}{2} = -gy + C.$$

When  $y = 0$ ,  $v = v_a$ , and hence  $C = v_a^2/2$ , and

$$v_a^2 - v^2 = 2gy.$$

When the particle is at  $B$ ,  $v = v_b$ ,  $y = h$ , and

$$v_a^2 - v_b^2 = 2gh.$$

(2) If the motion is from  $B$  toward  $A$ , let  $s = BP$ ; then

$$\frac{W}{g} v \frac{dv}{ds} = W \sin \theta.$$

Here  $s$  is increasing as  $y$  is decreasing, and hence  $\sin \theta = -dy/ds$ , so that

$$\frac{v}{g} \frac{dv}{ds} = -\frac{dy}{ds}.$$

Integrating, we find

$$\frac{v^2}{2} = -gy + C.$$

When  $y = h$ ,  $v = v_b$ , and hence  $C = v_b^2/2 + gh$ , and

$$\frac{v^2}{2} = -gy + \frac{v_b^2}{2} + gh.$$

When  $y = 0$ ,  $v = v_a$ , and  $v_a^2/2 = v_b^2/2 + gh$ , or

$$v_a^2 - v_b^2 = 2gh,$$

which is the same relation as when the particle moves from  $A$  to  $B$ .

This relation is independent of the form of the path, and holds in any case of motion in a vertical plane of a particle under the action of gravity and any other force which is at every point perpendicular to the path. As a special case, it applies to a particle moving under the action of gravity only, either in a straight line or a curve.

### PROBLEMS

299. A stone is thrown from the top of a building with a velocity of 100 ft./sec., and strikes the ground 80 ft. below the point of projection. With what velocity does it strike?

300. A piece of ice slips from a roof, starting at a point 10 ft. higher than the edge where it leaves the roof. Assuming that there is no friction, at what velocity does it leave the roof and with what velocity does it strike the ground 25 ft. below?

301. A small shot is projected horizontally from the edge of a table top 3 ft. from the floor with a velocity of 5 ft./sec. With what velocity does it

strike the floor? If released from rest at the edge of the table, with what velocity would it reach the floor? What is the vertical component of the velocity on striking the floor in the first case?

302. A small bead on a smooth vertical ring of radius  $r$  is projected from the lowest point of the ring with a velocity of such value that on reaching the highest point of the ring there is no pressure of the ring against the bead. Find the velocity of the bead at the highest point, at the lowest point. What is the pressure of the ring against the bead when at the lowest point?

303. A small heavy object is dropped from an airplane when it is moving horizontally at 100 ft./sec. at a height of 300 ft. above the earth's surface. With what velocity does it reach the surface of the earth? What is the vertical component of the velocity with which it strikes? What angle does the velocity make with the vertical?

**106. The Simple Pendulum.** Theoretically, a *simple pendulum* is a particle which is attached to one end of a weightless thread whose other end is attached to a fixed point, and which oscillates under the action of gravity.

The forces acting on the particle at any instant are  $W$ , the weight, and  $T$ , the tension (Fig. 228). Since  $T$  is always perpendicular to the path, the theorem of § 105 applies. Hence, if the particle starts from rest when the thread makes an angle  $\theta_0$  with the vertical, the velocity when the thread makes an angle  $\theta$  with the vertical, is given by the equation

$$(1) \quad v^2 = 2gl(\cos \theta - \cos \theta_0),$$

where  $l$  is the length of the thread. If  $s$  is measured from the lowest point of the path to any other point, and is counted positive toward the right, the velocity is

$$v = \pm \frac{ds}{dt} = \pm l \frac{d\theta}{dt},$$

the plus sign being used for motion to the right and the minus sign for motion to the left.

Equation (1) shows that, for given values of  $l$  and  $\theta_0$ , the numerical value of  $v$  depends only upon the numerical value of the angle  $\theta$ , and not upon the direction of motion nor upon the

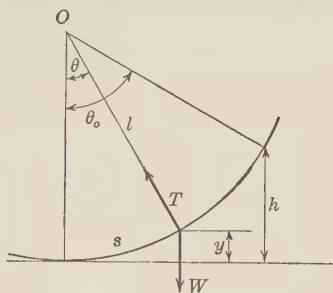


FIG. 228

sign of  $\theta$ . Hence the motion is periodic, the particle swinging between the positions determined by  $\theta_0$  and  $-\theta_0$ , and the time of a complete oscillation, that is, from any position to the same position with the same direction of motion, is four times the time required for  $\theta$  to change from 0 to  $\theta_0$ .

To find the time of vibration, let us replace  $v$  by  $l d\theta/dt$ ,  $\cos \theta$  by  $1 - 2 \sin^2 (\theta/2)$ , and  $\cos \theta_0$  by  $1 - 2 \sin^2 (\theta_0/2)$  in equation (1). The equation is then reduced to

$$(2) \quad l^2 \left( \frac{d\theta}{dt} \right)^2 = 4gl \left\{ \sin^2 \left( \frac{\theta_0}{2} \right) - \sin^2 \left( \frac{\theta}{2} \right) \right\}.$$

(a) AN APPROXIMATE SOLUTION. When the angle through which the particle swings is small,  $\sin (\theta_0/2)$  and  $\sin (\theta/2)$  may be replaced by  $\theta_0/2$  and  $\theta/2$ , respectively, and an approximate value of the time of oscillation is easily obtained. Then (2) becomes

$$\frac{d\theta}{dt} = \pm \sqrt{\frac{g}{l}} \sqrt{\theta_0^2 - \theta^2},$$

and if the motion from  $\theta = 0$  to  $\theta = \theta_0$  is considered, the plus sign must be used before the radical. Then

$$dt = \sqrt{\frac{l}{g}} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}},$$

whence

$$t = \sqrt{\frac{l}{g}} \sin^{-1} \frac{\theta}{\theta_0} + C.$$

Measuring  $t$  from the instant when  $\theta = 0$ , and calling  $\sin 0$  zero, the value of  $C$  is 0. Thus as  $\theta$  increases from 0 to  $\theta_0$ ,  $\sin^{-1} (\theta/\theta_0)$  must increase, since  $t$  increases, from 0 to  $\sin^{-1} 1$ , or  $\pi/2$ . Hence if  $t_1$  is the time of a complete oscillation,

$$\frac{t_1}{4} = \frac{\pi}{2} \sqrt{\frac{l}{g}},$$

or

$$t_1 = 2\pi \sqrt{\frac{l}{g}}.$$

(b) THE GENERAL SOLUTION. In equation (2), let us set  $\sin (\theta_0/2) = k$ , and  $\sin (\theta/2) = k \sin \phi$ . Then

$$\frac{1}{2} \cos \frac{\theta}{2} d\theta = k \cos \phi d\phi,$$



or

$$d\theta = \frac{2k \cos \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

and equation (2) is reduced to the form

$$\frac{2k \cos \phi \cdot l}{\sqrt{1 - k^2 \sin^2 \phi}} \frac{d\phi}{dt} = 2\sqrt{gl} k \sqrt{1 - \sin^2 \phi},$$

or

$$dt = \sqrt{\frac{l}{g}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

and

$$(3) \quad t_1 = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where  $t_1$  is the period of oscillation, and where the limits, 0 and  $\pi/2$ , of  $\phi$  correspond to the values 0 and  $\theta_0/2$  of  $\theta$ . The right-hand member of equation (3) can be integrated only by expanding the integrand by the binomial theorem,

$$\begin{aligned} (1 - k^2 \sin^2 \phi)^{-1/2} &= 1 + \frac{1}{2} k^2 \sin^2 \phi \\ &+ \frac{1 \cdot 3}{1 \cdot 2} \frac{1}{2^2} k^4 \sin^4 \phi + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{1}{2^3} k^6 \sin^6 \phi + \dots \end{aligned}$$

By a reduction formula of integration,

$$\int \sin^n \phi d\phi = -\frac{\cos \phi \sin^{n-1} \phi}{n} + \frac{n-1}{n} \int \sin^{n-2} \phi d\phi.$$

Substituting the limits 0 and  $\pi/2$ , we find that this reduces for even values of  $n$  to

$$\int_0^{\pi/2} \sin^n \phi d\phi = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} \phi d\phi = \frac{(n-1)(n-3) \dots 3 \cdot 1}{n(n-2) \dots 4 \cdot 2} \frac{\pi}{2}.$$

The value of  $t_1$  then is reduced to

$$t_1 = 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \dots \right\},$$

where  $k = \sin (\theta_0/2)$ .

### PROBLEMS

304. What length of simple pendulum is required for the pendulum to "beat" seconds, that is for the period of oscillation to be 2 seconds?

305. If a simple pendulum of length 3 ft. swings through an arc of  $40^\circ$ , what is the period of oscillation if  $g = 32.2$ ? What is the period for very small oscillations?

306. The period of a simple pendulum is 1 sec. when swinging through a very small arc. If the arc be increased to  $40^\circ$ , what will the period be?

307. A small weight is suspended by a light string 3 in. long; find approximately the time of a small oscillation.

308. A heavy weight is suspended by a rope 30 ft. long from the point of support to the center of the weight; how many small oscillations will it make in one minute.

309. A small lead sphere is suspended by a light thread which measures 16 ft. from the point of support to the center of the sphere. When set swinging through a small arc, it is found that it makes 100 oscillations in 7 min. 23 sec.; find the value of  $g$  at the place of observation.

**107. The Cycloidal Pendulum.** It has been found that the period of the simple pendulum depends upon the angle through which the pendulum oscillates. It is possible to construct a form on which the thread of the pendulum will wrap as it moves from the vertical position, in such a way that the period will be independent of the size of the oscillation. Such a form is that of a cycloid generated by a circle of radius one-fourth the length of the pendulum. We shall now prove this statement.

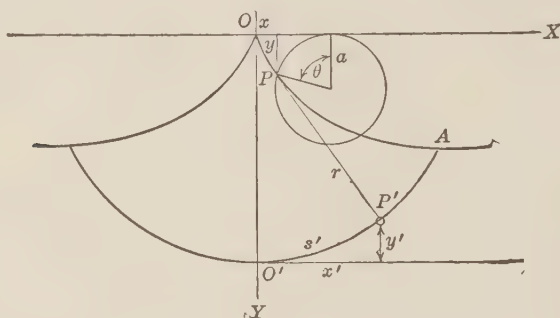


FIG. 229

Let a cycloid be generated by a point on a circle of radius  $a$ , rolling on the under side of a horizontal straight line (Fig. 229). If the cusp,  $O$ , of the cycloid is taken as origin, and if  $y$  is counted positive downward, the values of the coordinates of any point on the cycloid in terms of the angle turned through are

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

Then

$$dx = a(1 - \cos \theta)d\theta, \quad dy = a \sin \theta d\theta,$$

and

$$ds = (dx^2 + dy^2)^{1/2} = 2a \sin (\theta/2)d\theta.$$

Hence the length of the arc  $OP$  is

$$s = 2a \int_0^{\theta} \sin \frac{\theta}{2} d\theta = 4a \left( 1 - \cos \frac{\theta}{2} \right).$$

If  $\theta = \pi$ ,  $s = 4a$ , which is therefore the length of half of one arch of the cycloid. This is the length of the arc  $OA$  in Fig. 229; hence the arc  $PA$  is  $4a \cos (\theta/2)$ .

Now if a particle is suspended from  $O$  by a string of length  $4a$ , and is set swinging against a guide in the shape of the cycloid, when the string wraps on the guide from  $O$  to  $P$ , the free part,  $PP'$ , is equal to the arc  $PA$ , or  $4a \cos (\theta/2)$ .

From the equation

$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = \cotn \frac{\theta}{2},$$

it follows that the inclination of  $r$ , or  $PP'$ , to the vertical is  $\theta/2$ .

If the lowest point  $O'$  reached by the particle, is taken as origin, the coordinates of the position  $P'$  are

$$\begin{aligned} x' &= x + r \sin (\theta/2) \\ &= a(\theta - \sin \theta) + 4a \cos (\theta/2) \sin (\theta/2) = a(\theta + \sin \theta), \\ y' &= 4a - y - r \cos (\theta/2) \\ &= 4a - a + a \cos \theta - 4a \cos^2 (\theta/2) = a(1 - \cos \theta). \end{aligned}$$

The forces acting on the particle are the weight,  $W$ , and the tension in the string, which is normal to the path. The formula for the velocity obtained in § 105 therefore holds here. Hence if the particle is released from rest when at a height  $y_0'$  above  $O'$ , the velocity at the point  $P'$  is given by the equation

$$v^2 = 2g(y_0' - y').$$

$$\begin{aligned} \text{Now } v^2 &= (ds'/dt)^2 \text{ and } ds'^2 = dx'^2 + dy'^2 \\ &= a^2 \{ (1 + \cos \theta)^2 + \sin^2 \theta \} d\theta^2 = 4a^2 \cos^2 (\theta/2) d\theta^2. \end{aligned}$$

Therefore

$$ds' = 2a \cos (\theta/2) d\theta,$$

whence

$$2a \cos \frac{\theta}{2} \frac{d\theta}{dt} = \sqrt{2g} \sqrt{y_0' - y'} = \sqrt{2g} \left\{ 2a \sin^2 \frac{\theta_0}{2} - 2a \sin^2 \frac{\theta}{2} \right\}^{1/2},$$

where the plus sign is used for motion from  $O'$  to the right.

It follows that

$$dt = \sqrt{\frac{a}{g}} \cdot \frac{\cos (\theta/2) d\theta}{\{\sin^2 (\theta_0/2) - \sin^2 (\theta/2)\}^{1/2}},$$

and if  $t_1$  is the period of vibration,

$$\int_0^{t_1} dt = 4 \sqrt{\frac{a}{g}} \cdot \int_0^{\theta_0} \frac{\cos (\theta/2) d\theta}{\{\sin^2 (\theta_0/2) - \sin^2 (\theta/2)\}^{1/2}},$$

or

$$t_1 = 8 \sqrt{\frac{a}{g}} \sin^{-1} \left[ \frac{\sin (\theta/2)}{\sin (\theta_0/2)} \right]_0^{\theta_0} = 8 \sqrt{\frac{a}{g}} \frac{\pi}{2}.$$

Since the length,  $l$ , of the pendulum is  $4a$ , this becomes

$$t_1 = 2\pi \sqrt{\frac{l}{g}}.$$

This result is independent of the size of the angle through which the pendulum swings.

**108. The Conical Pendulum.** Suppose that a particle of weight  $W$  travels in a smooth grooved circular path in a horizontal plane with constant speed  $v$  (Fig. 230). The resultant force that must act upon it in order for it to maintain this motion is  $(W/g)(v^2/r)$  directed toward the center of the path (§ 101).

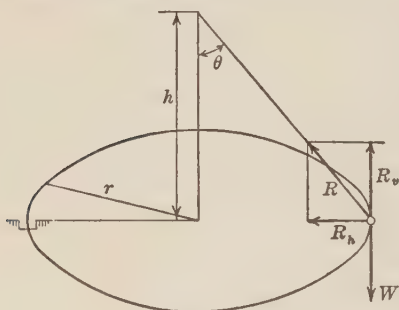


FIG. 230

It would keep this motion if the only forces acting on it were the weight and the reaction of the track, for if the track is smooth there would be no tangential force and hence the speed would not change, and the track would prevent the particle

from leaving the circle. Let the reaction of the track upon the particle be resolved into a vertical component,  $R_v$ , and a horizontal component,  $R_h$ . Since there is no vertical acceleration, the forces  $R_v$  and  $W$  must be equal. Also  $R_h$  must have the value  $Wv^2/(gr)$ . The reaction  $R$  is therefore a force in a vertical plane through the center of the track and making an angle

$\theta = \tan^{-1} (R_h/R_v)$ , or  $\tan^{-1} [v^2/(gr)]$ , with the vertical. This force could be furnished by a weightless string, one end of which is attached to the particle and the other to a fixed point directly above the center of the circular path and at a distance  $h = r/\tan \theta$ , or  $gr^2/v^2$ , from the plane of motion. If the string were so attached, the grooved track could be removed without altering the motion. If  $\omega$  is the angle passed through per unit time by the radius of the circle which passes through the particle,  $v = r\omega$ , and the relation  $h = gr^2/v^2$  may be written in the form  $h = g/\omega^2$ .

The quantity  $\omega$  is called the **angular velocity** of the particle with respect to the center of the path.

Hence it is possible to attach a particle to one end of a weightless string, the other end of which is attached to a fixed point, and have the particle describe a circle in a horizontal plane at a distance  $h$  below the point of support and with constant angular velocity  $\omega$  with respect to the projection of the point of support on the plane, provided only that the relation  $h = g/\omega^2$  is satisfied. A particle moving in this way is called a **conical pendulum** (Fig. 231).

As the angular velocity increases, the distance of the plane of motion below the point of support decreases. The value of  $h$  depends only upon the angular velocity of the particle.

A practical application of this principle is seen in the governor of a steam engine (Fig. 232). Weights attached to rods are pinned to a vertical shaft which is geared to the engine shaft. As the shaft rotates, the weights rise or fall according to the speed, and in so doing operate a valve which controls the amount of steam entering the cylinder.

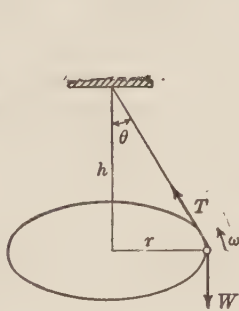


FIG. 231

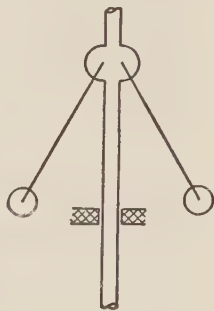


FIG. 232

### PROBLEMS

310. Show that the path traced by the particle in the cycloidal pendulum, § 107, is a cycloid generated by a circle of radius  $a$  rolling on a horizontal line tangent to the cycloidal guides.

311. Derive the equations of the cycloid of Fig. 233, and show that a particle placed on a smooth guide of this form and displaced from the position of rest would vibrate with a period of  $4\pi\sqrt{a/g}$ .

312. A conical pendulum is making 120 r.p.m. How far below the point of support is the plane of motion? If the speed is changed to 60 r.p.m., what does the distance become? If the string is 4 ft. long and the attached weight is 2 lbs., what is the tension in the string in each case?

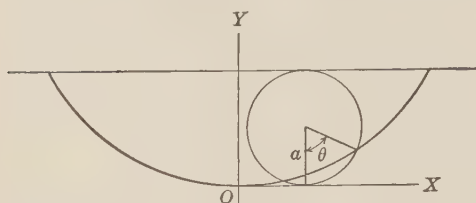


FIG. 233

Assuming the rods to be weightless and the weights to be concentrated at points, find the number of r.p.m. that will cause the rods to make angles of  $30^\circ$  with the vertical. What are then the tensions in the rods?

314. Fig. 235 represents a vertical section of a smooth circular banked track. If a particle travels around this track in a horizontal circle of radius  $r$  under the action of its own weight and the reaction of the track, what relation exists between the radius,  $r$ , the angle of elevation,  $\theta$ , and the speed,  $v$ , of the particle? From the result compute the best angle of elevation of a circular running track of radius 50 ft. for a speed of 10 mi./hr.

315. The outer rail of a railway track is raised a distance  $d$  above the inner on a curve of radius  $r$ . If the elevation is

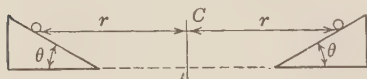


FIG. 235

such that the pressure of the car on the track is normal to the track when the speed of the car is  $v$ , and if  $b$  is the distance between the rails, what relation exists between  $b$ ,  $d$ ,  $r$ , and  $v$ ?

**109. Relative Displacement of Two Moving Points.** Suppose two points,  $P$  and  $Q$ , to move in any way along two curves in a plane (Fig. 236). If  $P$  moves from  $A$  to  $A'$  at the same time that  $Q$  moves from  $B$  to  $B'$ , the difference of the vectors  $A'B'$  and  $AB$

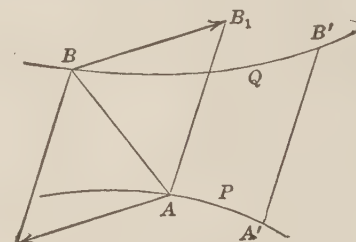


FIG. 236

of support is the plane of motion? If the speed is changed to 60 r.p.m., what does the distance become? If the string is 4 ft. long and the attached weight is 2 lbs., what is the tension in the string in each case?

313. The rods of Fig. 234 are pivoted at the ends and are free to turn.

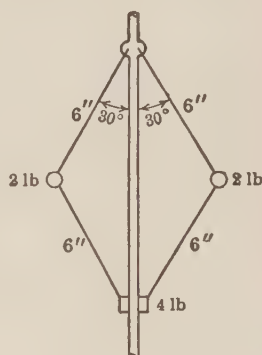


FIG. 234



is called the *displacement* of  $Q$  relative to  $P$ . That is, if a vector  $AB_1$ , equal to the vector  $A'B'$ , is laid off from  $A$ , the vector  $BB_1$  represents the displacement of  $Q$  relative to  $P$ . The reverse of this vector,  $AA_1$ , laid off from  $A$ , represents the displacement of  $P$  relative to  $Q$ .

**110. Relative Velocity of Two Moving Points.** Let  $P$  and  $Q$  be two moving points (Fig. 237). At a given instant let  $v_p$  and  $v_q$  be the corresponding velocities of  $P$  and  $Q$ . In Fig. 237  $v_p = AC$ , and  $v_q = BD$ . The vector difference,  $v_q - v_p$ , is called the velocity of  $Q$  relative to  $P$ . Thus, a vector  $AE$ , equal to the vector  $v_q$ , is laid off from  $A$ ; the vector  $CE$ , or  $AF$ , either of which is equal to the vector  $v_q - v_p$ , then represents the velocity of  $Q$  relative to  $P$ .

It is useful to note that if both points had, in addition to their actual velocities, a velocity equal and opposite to that of  $P$ , the point  $P$  would be brought to rest and the resultant velocity of  $Q$  would be the velocity of  $Q$  relative to  $P$ .

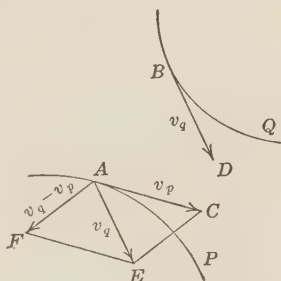


FIG. 237

### PROBLEMS

316. Two automobiles,  $A$  and  $B$ , move respectively 5 mi. east and 10 mi. southwest from the same point. Find the magnitude and direction of the displacement of  $B$  relative to  $A$ .

317. If the automobiles of the preceding problem are traveling at a given instant at 30 mi./hr. and 40 mi./hr. respectively in the directions given, find the velocity of  $B$  relative to  $A$ .

318. A car is moving horizontally at 10 mi./hr. Rain falling vertically strikes the car window at an angle of  $30^\circ$  with the vertical. Find the velocity of the rain drops. If the car changes its speed to 25 mi./hr., what angle will the drops striking the window make with the vertical?

319. An airplane headed due north is changed by a steady 30 mi./hr. wind from the southwest to a direction  $N. 12^\circ E.$  What is its velocity relative to the air, and what relative to the earth?

320. An aviator can fly at 100 mi./hr. in still air. With a wind blowing at 30 mi./hr. from the west, what time will be required for the aviator to fly due north 100 mi.?

321. Two automobilists are traveling on two straight roads that intersect. At the same distance from the intersection, each car appears to an observer in the other car to be moving directly toward him. Compare their speeds.



increases. It is also clear from the figure that there may be a forward component of the force due to the wind even when the velocity of the boat is greater than the velocity of the wind. The resistance of the ice being small, the velocity of the boat can be made to exceed the velocity of the wind for high wind velocities.

### PROBLEMS

322. Assuming that there is no frictional resistance to the forward motion of the boat, show that the velocity of the boat will increase until the end  $B$  of the vector  $AB$ , representing the reversed velocity of the boat in Fig. 238, falls on the line  $SL$  which represents the sail. Hence show that if there were no friction, the maximum velocity of the boat would be

$$v_b = v_w \sin(\alpha + \beta) / \sin \alpha,$$

where  $\alpha$  and  $\beta$  are respectively the angles which the sail and the direction of the wind's velocity make with the direction of the boat. (See Fig. 238.) If  $\alpha = 30^\circ$  and  $\beta = 40^\circ$ , show that the maximum velocity that the boat could attain with no friction would be about 1.88 times the velocity of the wind.

323. With a given angle  $\alpha$  and direction of boat's motion, prove that if there were no friction, the greatest velocity of the boat would be attained when the direction of the boat makes with the direction of the wind an angle equal to the complement of  $\alpha$ , and show that this maximum velocity is  $v_w \csc \alpha$ . What does this maximum velocity become when  $\alpha = 30^\circ$ ?

324. An ice boat has a sail area of 45 sq. ft. The angles  $\alpha$  and  $\beta$  of Fig. 238 are  $25^\circ$  and  $40^\circ$ , respectively. The resistance offered by the ice to the motion of the boat is 20 lbs. If the pressure of the wind striking normally with velocity  $v$  on a plane surface at rest is  $kv^2$  and is 1 lb. per sq. ft. when  $v = 15$  mi./hr., show that the force of the wind driving the boat forward is  $P_f = 0.08452 v_n^2$  and that the maximum velocity of the boat is attained when  $v_n = 15.38$  mi./hr. Find the maximum velocity that the boat can attain when the wind velocity is 20 mi./hr., 35 mi./hr.

### 112. Angular Velocity and Angular Acceleration of a Particle with Respect to a Point.

When a particle travels along any plane curve, the rate at which a line joining the particle and a fixed point in the plane of motion is turning at any instant is called the **angular velocity** of the particle with respect to the

fixed point at that instant. Thus, in Fig. 239, if  $OA$  is a fixed line,  $\theta$  the angle which a line joining  $O$  and the moving point  $P$  makes

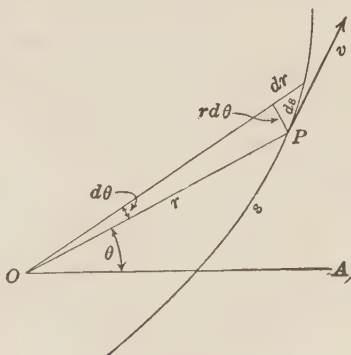


FIG. 239

with  $OA$ , the value of  $d\theta/dt$  at any instant is the angular velocity of  $P$  with respect to  $OA$  at that instant. Calling the angular velocity  $\omega$ , we may then write

$$\omega = \frac{d\theta}{dt}.$$

The *angular acceleration* of  $P$  with respect to  $O$  is defined to be the rate of change of the angular velocity of  $P$  with respect to  $O$ . Calling the angular acceleration  $\alpha$ , we then have

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Since we may write

$$\frac{d\omega}{dt} = \frac{d\omega}{d\theta} \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta},$$

the value of  $\alpha$  may also be expressed in the form

$$\alpha = \omega \frac{d\omega}{d\theta}.$$

From the definitions given it is evident that, in a given motion of a particle, the values of the angular velocity and angular acceleration of the particle with respect to the point depend upon the point to which they are referred.

**113. Tangential and Normal Components of Acceleration of a Moving Particle in Terms of Polar Coordinates.** Referring again to Fig. 239, let  $r$  and  $\theta$  be the polar coordinates of the particle at any instant and  $s$  the length of arc measured from some fixed point of the path to  $P$ . Then the numerical value of the velocity of the particle is  $v = \pm ds/dt$  (§ 101). If, as the particle moves, the value of  $s$  is increasing, the plus sign is used, and if  $s$  is decreasing the minus sign is used. In any case the vector representing the velocity of the particle in any position is tangent to the path and in the direction of motion of the particle at the instant in question.

From the figure we have

$$ds^2 = (r d\theta)^2 + (dr)^2,$$

and hence

$$v^2 = \left( r \frac{d\theta}{dt} \right)^2 + \left( \frac{dr}{dt} \right)^2.$$

If  $r$  is expressed as a function of  $\theta$ , it is convenient to write

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \omega \frac{dr}{d\theta},$$

from which we find

$$v^2 = \omega^2 \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}.$$

Differentiating both sides of this equation with respect to  $t$  and simplifying, we obtain the tangential component of the acceleration

$$a_t = \frac{dv}{dt} = \frac{\omega^2 \frac{dr}{d\theta} \left( r + \frac{d^2r}{d\theta^2} \right) + \alpha \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]}{\left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]^{1/2}}.$$

The normal component of the acceleration is

$$a_n = \frac{v^2}{\rho} = \frac{\omega^2 \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]}{\rho},$$

where  $\rho$  is the radius of curvature of the path of the particle at the point in question. The value of  $\rho$  is given by the formula

$$\rho = \frac{\left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]^{3/2}}{r^2 + 2 \left( \frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2}}.$$

In computing the value of  $\rho$  any convenient origin may be used, that is, the  $r$  and  $\theta$  are not necessarily the same as in Fig. 239.

For the case of a particle moving in a circle of radius  $r$ , if the motion is considered with reference to the center of the circle,  $r$  is constant,  $dr/d\theta$  and  $d^2r/d\theta^2$  are both zero, and the formulas for  $v$ ,  $a_t$ , and  $a_n$  are reduced to

$$v = r\omega, \quad a_t = r\alpha, \quad a_n = r\omega^2.$$



**114. Radial and Transverse Components of Acceleration.** In Fig. 240 the velocity  $v$  is resolved into components  $PM$  and  $PN$ ,

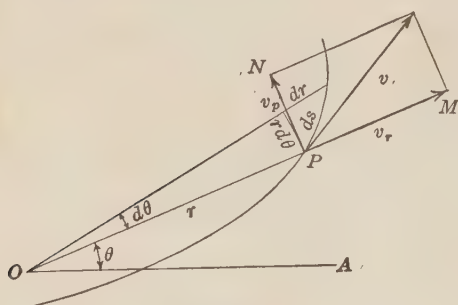


FIG. 240

along the radius and perpendicular to the radius, respectively. The former is denoted by  $v_r$  and the latter by  $v_p$ . We have then

$$v_r = v \cos \psi, \quad v_p = v \sin \psi.$$

$$\text{But } \cos \psi = dr/ds, \sin \psi = r d\theta/ds, \text{ and } v = ds/dt.$$

Hence

$$v_r = \frac{dr}{dt}, \quad v_p = r \frac{d\theta}{dt}.$$

To find the component of the acceleration along  $OP$  it must be noted that since  $r$  is a variable direction, the change in the component of velocity along  $OP$  is due to a change in both  $v_r$  and  $v_p$  (Fig. 241). Thus when the particle moves from  $P$  to  $P'$ , the gain in the velocity component in the direction  $OP$  is

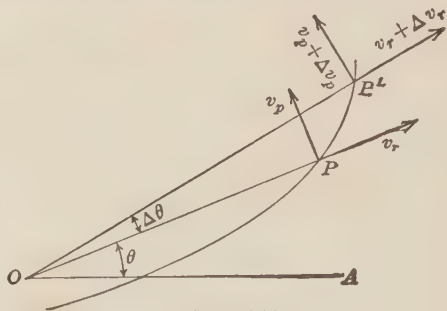


FIG. 241

$$(v_r + \Delta v_r) \cos \Delta\theta - v_r - (v_p + \Delta v_p) \sin \Delta\theta.$$

The limit of the ratio of this quantity to  $\Delta t$ , the time required for the particle to move from  $P$  to  $P'$ , is the component of the acceleration along  $OP$ . The ratio may be written in the form

$$\frac{-v_r(1 - \cos \Delta\theta) + \Delta v_r \cos \Delta\theta}{\Delta t} - \frac{(v_p + \Delta v_p) \sin \Delta\theta}{\Delta t}.$$

The first fraction approaches the limit  $dv_r/dt$  and the second approaches the limit  $v_p d\theta/dt$ . (Compare the derivations for  $a_t$  and  $a_n$  in § 101.) Hence, calling the acceleration along the



radius  $a_r$ , we have  $a_r = dv_r/dt - v_p d\theta/dt$ , or, since  $v_r = dr/dt$  and  $v_p = r d\theta/dt$ ,

$$a_r = \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2.$$

To find the transverse acceleration,  $a_p$ , perpendicular to  $r$ , we have to find the limit of

$$\frac{(v_p + \Delta v_p) \cos \Delta\theta - v_p + (v_r + \Delta v_r) \sin \Delta\theta}{\Delta t}.$$

This is seen to reduce to  $dv_p/dt + v_r d\theta/dt$ . Substituting for  $v_p$  and  $v_r$  their values, we have

$$a_p = \frac{d}{dt} \left( r \frac{d\theta}{dt} \right) + \frac{dr}{dt} \frac{d\theta}{dt} = r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt}.$$

An equivalent and more compact form is

$$a_p = \frac{1}{r} \frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right).$$

### 115. Angular Velocity and Angular Acceleration of a Particle with Respect to an Axis.

When a particle moves along any path in space, its angular velocity with respect to an axis is defined to be the rate of change of direction of a line which is perpendicular to the given axis, intersects it, and passes through the particle. Thus (Fig. 242) if  $P$  is a moving particle,

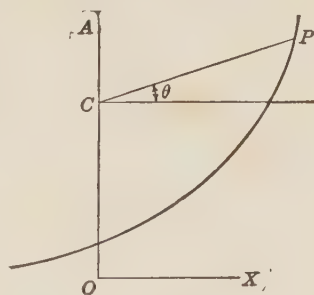


FIG. 242

$OA$  a fixed axis,  $CP$  a line intersecting  $OA$  at right angles, and  $\theta$  the angle which  $CP$  makes with a fixed direction,  $OX$ , perpendicular to  $OA$ , then the value of the angular velocity,  $\omega$ , of  $P$  with respect to  $OA$ , is  $\omega = d\theta/dt$ .

In the same way, the angular acceleration of  $P$  with respect to  $OA$  is  $\alpha = d^2\theta/dt^2$ .

## PROBLEMS

325. A particle traveling around a circle of radius  $c$  has at any instant an angular velocity  $\omega$  with respect to the center of the circle. Show that the angular velocity of the particle with respect to any point on the circumference is at that instant equal to  $\omega/2$ . Using the equation of the circle referred to  $O$  as origin (Fig. 243),  $r = 2c \cos \theta$ , show that  $v = cd\phi/dt = c\omega$ ,  $a_t = cd\omega/dt$ , and  $a_n = v^2/c$ .

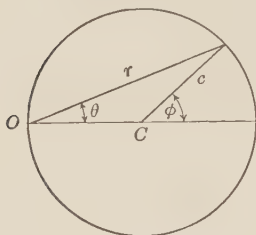


FIG. 243

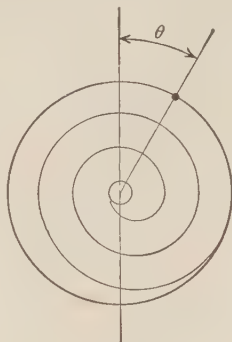


FIG. 244

326. A point on the rim of a balance wheel of a watch vibrates through an angle of  $120^\circ$ , having an angular acceleration with respect to the center which varies directly as the angular displacement of the point from its central position, but opposite in sign. If two complete vibrations are made per second, find the maximum angular velocity and the maximum angular acceleration.

(SUGGESTION. Since the angular acceleration is  $\omega d\omega/d\theta$  (§ 112), we may write  $\omega d\omega = -k\theta d\theta$  (see Fig. 244). Integrate between the limits  $\pi/3$  and  $\theta$  for  $\theta$  to obtain the relation between  $\omega$  and  $\theta$  for any position. Then replace  $\omega$  by  $d\theta/dt$  and integrate again between the limits 0 and  $1/8$  for  $t$ , for the determination of  $k$ .)

327. Show that if a particle has constant angular acceleration  $\alpha$ , with respect to a point, the following formulas hold:

$$\alpha = (\omega - \omega_0)/t, \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2, \quad \omega^2 - \omega_0^2 = 2\alpha\theta,$$

where  $\theta$  is the angle turned through in the time  $t$  by the line joining the particle and the given fixed point,  $\omega_0$  the angular velocity of the particle with respect to the point when  $t = 0$ , and  $\omega$  the angular velocity at the time  $t$ .

328. Consider a point on the rim of a flywheel that is making 120 revolutions per minute at a given instant. If it has a constant (negative) angular acceleration which brings it to rest in 40 revolutions, find the time required to come to rest.

329. A small, straight, smooth rod is pivoted at one end,  $O$ , and rotates in a horizontal plane with constant angular velocity  $\omega$ . A small bead of mass  $m$  is placed on the rod and has zero velocity along the rod when the distance from the pivot is  $c$ . Show that when the rod has turned through an additional angle  $\theta$ , the bead will be at a distance  $r = (c/2)(e^\theta + e^{-\theta})$  from  $O$ , and that its

velocity along the rod will be  $\omega(r^2 - c^2)^{1/2}$ . Show also that the pressure of the rod against the bead will be  $P = mc\omega^2(e^\theta - e^{-\theta})$ .

(SUGGESTION. Since the rod is smooth, there will be no force exerted on the bead along the rod. Hence  $a_r = 0$ , or  $d^2r/dt^2 = r\omega^2$ . Write  $d^2r/dt^2$  as  $dv_r/dt = (dv_r/dr)(dr/dt) = v_r dv_r/dr$  and integrate to find  $v_r$ . In the result replace  $v_r$  by  $dr/dt$  and integrate again. To obtain  $P$ , equate the force perpendicular to the rod to the product of  $m$  and  $a_p$ ).

330. A moving particle has at time  $t$  the coordinates  $x = r \cos ct$ ,  $y = r \sin ct$ ,  $z = kt$ , where  $r$ ,  $c$ , and  $k$  are constant. Show that if the positive direction of rotation about the  $z$  axis is from  $OX$  to  $OY$ , the angular velocity of the particle with respect to  $OZ$  is  $c$ .

331. Show that if the positive direction of rotation about the  $z$  axis is from  $OX$  to  $OY$ , the angular velocity with respect to the  $z$  axis of a particle whose coordinates at any instant are  $(x, y, z)$  is

$$\omega_z = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2 + y^2},$$

and derive corresponding formulas for the angular velocities of the particle with respect to the other coordinate axes.

(SUGGESTION. Note from Fig. 245 that  $\tan \theta = y/x$  and that the value of  $\omega_z$  is  $d\theta/dt$ .)

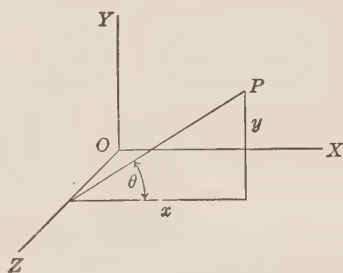


FIG. 245

**116. Angular Velocity and Angular Acceleration of a Body Having Plane Motion.** A body is said to have *plane motion* when each

point of the body moves in a fixed plane. All points of the body then move in the same plane or in parallel planes. That one of these parallel planes which passes through the center of mass will be called the *plane of motion*. Clearly all lines lying in the plane of motion, or in a plane parallel to it, are at any instant turning at the same rate. The rate at which any line of the body in the plane of motion is turning at any instant is called the *angular velocity* of the body at that instant. The rate of change of the angular velocity is called the *angular acceleration* of the body.

If the angular velocity of a body in plane motion is zero, the body has *pure translation*. The particles of a body in pure translation do not necessarily move in straight lines.

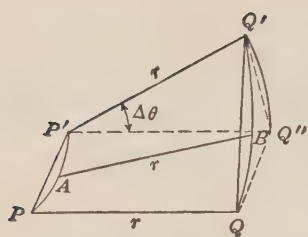


FIG. 246

**117. Relations between the Velocities and Accelerations of Two Connected Points Moving in the Same Plane.** Let two points,  $A$  and  $B$ , at a constant distance,  $r$ , apart, move in the same plane. Let the point  $A$  move from  $P$  to  $P'$  as  $B$  moves from  $Q$  to  $Q'$  (Fig. 246). When at  $P$  and

$Q$  respectively let the velocities of  $A$  and  $B$  be  $v_a$  and  $v_b$ . Then, by § 101,

$$v_a = \lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta t}, \quad v_b = \lim_{\Delta t \rightarrow 0} \frac{QQ'}{\Delta t}.$$

Now the displacement of  $AB$  from  $PQ$  to  $P'Q'$  may be made by moving  $AB$  from the position  $PQ$  to the position  $P'Q''$  without any change in direction, and then turning  $AB$  through an angle  $\Delta\theta$  about  $P'$  until  $B$  falls at  $Q'$ . The vector  $QQ'$  may then be regarded as the sum of the vectors  $QQ''$  and  $Q''Q'$ . Assuming the displacements to be infinitesimal, the value of  $Q''Q'$  may be replaced by  $r\Delta\theta$ . We may then write

$$\lim_{\Delta t \rightarrow 0} \frac{QQ'}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta t} + r \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t},$$

in which the vector

$$r \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t}$$

is perpendicular to  $r$ . The vector equation then may be written in the form

$$v_b = v_a + r\omega,$$

in which the vector  $r\omega$  is perpendicular to  $r$ , and  $\omega = d\theta/dt$  is the angular velocity of  $AB$ . The relation between  $v_a$ ,  $v_b$ , and  $\omega$  is shown in Fig. 247.

The *acceleration* of  $B$  is the rate of change of the velocity of  $B$ , and since the velocity of  $B$  is the vector sum of the velocity of  $A$  and the vector  $r\omega$ ,

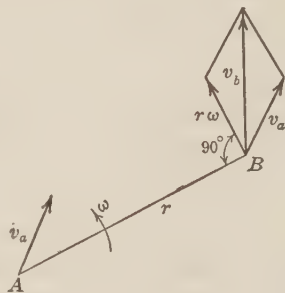


FIG. 247

perpendicular to  $r$ , the acceleration of  $B$  is the vector sum of the vectors which represent the rate of change of  $v_a$  and of  $r\omega$ . Now the rate of change of  $v_a$  is the acceleration of  $A$ , or  $a_a$ . Moreover,  $r\omega$  represents the velocity of a point moving in a circle of radius  $r$  and angular velocity  $\omega$ . The rate of change of this velocity has been found to be made up of the two vector components,  $r\alpha$ , perpendicular to  $r$ , and  $r\omega^2$  directed toward the center of the circle (§ 113), in which  $\alpha$  is the angular acceleration of the moving point with respect to the center of the circle. Hence the acceleration  $a_b$  of  $B$  is composed of the three

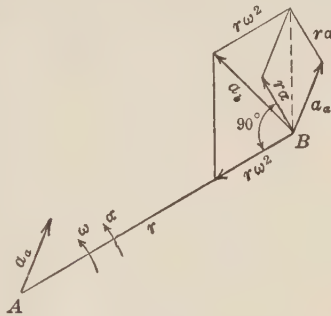


FIG. 248

components, (1) the acceleration of  $A$ , (2) a vector  $r\alpha$  perpendicular to  $r$ , (3) a vector  $r\omega^2$ , directed along  $BA$  toward  $A$ .

The relation between  $a_b$ ,  $a_a$ ,  $\omega$ , and  $\alpha$  is shown in Fig. 248, in which  $\omega$  and  $\alpha$  are respectively the angular velocity and angular acceleration of the line  $AB$  at the given instant.

## PROBLEMS

332. A wheel of radius  $r$  rolls without slipping along a straight line. Show that if  $v_c$  and  $a_c$ , respectively, are the velocity and the acceleration of the center of the wheel, the values of the angular velocity and angular acceleration of the wheel are  $\omega = v_c/r$  and  $\alpha = a_c/r$ , respectively. Then show how to find the velocity and acceleration of any point of the wheel. Note particularly the velocity and acceleration of the point of the wheel in contact with the line.

333. A wheel of radius 2 ft. rolls without slipping, in a vertical plane, along a horizontal straight line. At a given instant the velocity and acceleration of the center are respectively 5 ft./sec. and 4 ft./sec.<sup>2</sup>. Find the velocity and acceleration at that instant of the highest point of the wheel, of the lowest point, and of a point on the rim of the wheel to the rear of and 30° above the horizontal line through the center of the wheel.

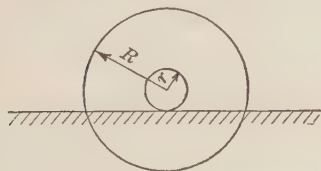
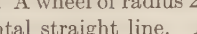


FIG. 249

334. A disk of radius  $R$  has an axle of radius  $r$  which rolls on a straight track (Fig. 249). If the velocity and acceleration of the center are  $v$  and  $a$  respectively, find the

velocity and acceleration of the highest point of the disk, of the lowest point, and of the point of the rim to the rear and  $\theta$  radians above the horizontal line through the center of the disk.



### 118. Instantaneous Axis of Rotation of a Body in Plane Motion.

Let  $A$ , with velocity  $v_a$ , be any point in a body in plane motion (Fig. 250). Through  $A$  draw a line perpendicular to the vector  $v_a$  and parallel to the plane of motion. The velocity of any point of the body on this line, distant  $r$  from  $A$ , is composed of two vectors,  $v_a$  and  $r\omega$ , both perpendicular to the line, where  $\omega$  is the angular velocity of the body (§ 117).

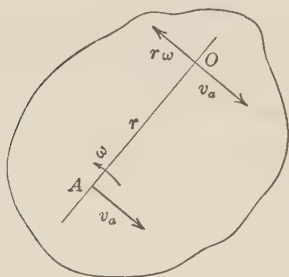


FIG. 250

On one side of  $A$  the vectors act in the same direction and on the other side in opposite directions. It is therefore clearly possible to find a point of the body, or of space thought of as moving with the body, on the line at such a distance  $r$  from  $A$  that  $r\omega = v_a$ , and such that the vectors  $v_a$  and  $r\omega$  are in opposite directions.

If  $O$  is this point, then the velocity of  $O$  is  $v_a - r\omega = 0$ . Clearly there is only one point on the line that fulfills this condition. Moreover, if any other point of the body in the plane through  $O$ , parallel to the plane of motion, is chosen, its velocity is  $r\omega$ , where  $r$  is the distance of that point from  $O$  (§ 117). This cannot be zero since  $\omega$  is not zero. There is therefore only one point of the body in a plane parallel to the plane of motion, that has zero velocity. From the definition of plane motion, it follows that all points of the body which lie in a line perpendicular to the plane of motion have the same velocity, and hence all points of the body that lie in a line through  $O$  and perpendicular to the plane of motion have zero velocity at the given instant. This line is called the **instantaneous axis of rotation**.

Hence, any plane motion may be interpreted as a rotation, the special case of pure translation corresponding to the value  $\omega = 0$ , in which case the instantaneous axis is "at infinity."

If we are dealing with a body which may be thought of as in one plane, the point  $O$  of this discussion is called the **instantaneous center**. The locus of the instantaneous centers is called the **centrode**.

Another method of locating the instantaneous axis is sometimes



more convenient. If the directions of motion of two points of the body in the plane of motion are known, the instantaneous center may be located by drawing through the two points lines in the plane of motion perpendicular to the velocities of these points respectively (Fig. 251). The intersection of these lines is a point in the instantaneous axis, for it has been shown above that if through any point of the body a line is drawn perpendicular to the velocity of that point, in a plane parallel to the plane of motion, this line cuts the instantaneous axis.

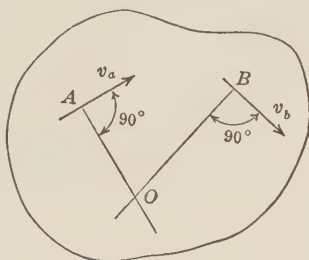


FIG. 251

## PROBLEMS

335. A thin straight rod moves in a vertical plane with one end against a vertical wall and the other against a horizontal floor. Show how to locate the instantaneous center for any position, and show that as the rod moves, the instantaneous center moves in a circle with radius equal to the length of the rod.

336. A rectangular sheet of cardboard moves in one plane so as to keep the ends of one edge of length  $a$  against two axes at right angles to each other (Fig. 252). Show that as the edge  $a$  moves from coincidence with one axis to coincidence with the other, the instantaneous center traces out a quadrant of a circle of radius  $a$  in space (the *space centrode*), and traces a semicircle of diameter  $a$  on the cardboard (the *body centrode*).

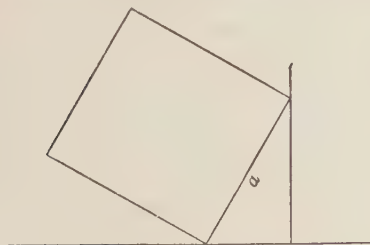


FIG. 252

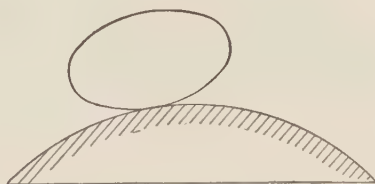


FIG. 253

337. Show that if a body having plane motion is rolling without slipping upon a fixed surface, the point of contact of the surfaces lies on the instantaneous axis of the rolling body at each instant.

(SUGGESTION. Show that the point of the rolling body can have no component of velocity along the tangent to the fixed surface, and also that it can have no component of velocity along the normal to the fixed surface at the point of contact (Fig. 253).)

338. In Fig. 254,  $AB$  represents the connecting rod of a stationary engine. Show that the construction indicated locates the instantaneous center,  $O$ , of the connecting rod for the position shown, and sketch the centrode as  $B$  moves through  $180^\circ$  from  $C$  to  $C'$ .

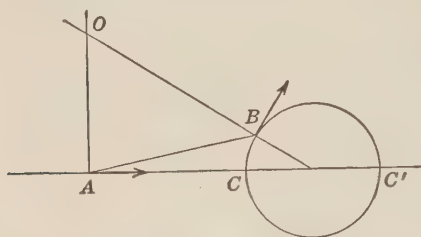


FIG. 254

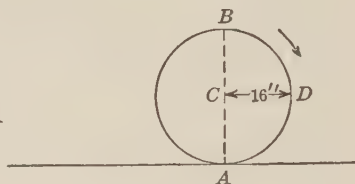


FIG. 255

339. The wheel of Fig. 255 is making 4 revolutions per second but the center is moving forward at the rate of only 20 ft./sec. Locate the instantaneous center and find the velocities of the points  $A$ ,  $B$ , and  $D$  on the rim of the wheel.

340. The wheel of Fig. 256 rolls on the axle along the straight line. The radius of the wheel is 20 inches and that of the axle is 8 inches. Sketch the path of the point  $A$ , 4 inches from the center, the point  $B$ , on the rim of the axle, and the point  $C$ , on the rim of the wheel, during one complete revolution.

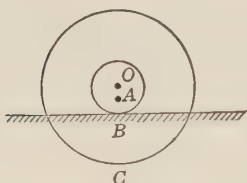


FIG. 256

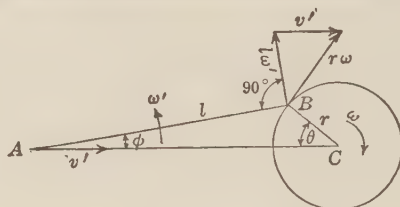


FIG. 257

341. In Fig. 257,  $AB$  represents the connecting rod of a stationary engine. In the position shown, the angular velocity of the crank pin,  $B$ , is  $\omega$ , the angular velocity of the connecting rod is  $\omega'$ , and the velocity of the cross-head,  $A$ , is  $v'$ . Show that  $\omega$ ,  $\omega'$ , and  $v'$  are connected as shown in the triangle, where  $r$  is the

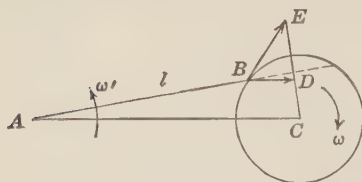


FIG. 258

radius of the crank-pin circle and  $l$  is the length of the connecting rod. Hence show how when one of the quantities,  $\omega$ ,  $\omega'$ , and  $v'$ , are given for a given position of the connecting rod, the other two may be determined graphically. Assuming  $r = 1$  ft.,  $l = 5$  ft., and  $\omega = 2\pi$  rad./sec., find by graphical construction the values of  $\omega'$  and  $v'$  for  $\theta = 0, 30^\circ, 90^\circ$ , and  $180^\circ$ .

342. In Fig. 258,  $CE$  is perpendicular to  $AB$ , and  $BD$  is parallel to  $AC$ . Show that if  $BE$ , tangent to the crank-pin circle, is taken as a measure of the

velocity of the crank pin, then  $BD$  represents  $v'$ , the velocity of the crosshead, and  $DE$  the value of  $l\omega'$ .

343. From the triangle of velocities in Fig. 257, derive the following relations

$$v' = \frac{r\omega \sin(\theta + \phi)}{\cos \phi}, \quad \omega' = \frac{r\omega \cos \theta}{l \cos \phi}.$$

Check the values found by graphical construction in Problem 341 by these formulas.

344. Assuming that the angular velocity,  $\omega$ , of the crank pin is constant, show that the angular velocity,  $\omega'$ , the angular acceleration,  $\alpha'$ , of the connecting rod, the acceleration,  $a'$ , of the crosshead, and the angular velocity  $\omega$  are connected as shown in the quadrilateral of Fig. 259. Sketch the figure for  $\theta = 0, 90^\circ, 180^\circ$ .

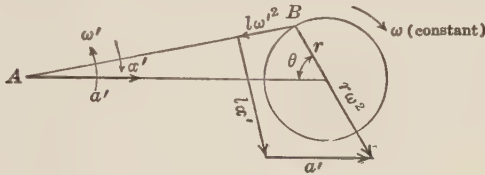


FIG. 259

With  $l = 5$  ft.,  $r = 1$  ft.,  $\omega = 2\pi$  rad./sec., find by graphical construction, the values of  $\alpha'$  and  $a'$  when  $\theta = 30^\circ$ . (Note the assumed direction of  $\alpha'$ .)

345. By projection of the sides of the quadrilateral of Fig. 259 upon the line of the connecting rod, and upon a perpendicular to the line  $AC$  respectively, show that the following formulas are obtained:

$$a' = \frac{l\omega'^2 + r\omega^2 \cos(\theta + \phi)}{\cos \phi}, \quad \alpha' = \frac{r\omega^2 \sin \theta}{l \cos \phi} - \omega'^2 \tan \phi.$$

346. Assuming  $\omega$  to be constant, show by the formulas and also by graphical construction that  $a'$  is positive when the connecting rod is tangent to the crank-pin circle for  $0 < \theta < 90^\circ$ , and is negative when  $\theta = 90^\circ$ , and that it remains negative until  $\theta = 180^\circ$ . Hence show that the maximum value of the velocity of the crosshead occurs at some intermediate point between the positions where the connecting rod is tangent to the crank-pin circle and where  $\theta = 90^\circ$ . When  $l = 4r$ , show by graphical construction that  $a'$  is zero very near to the position where the connecting rod is tangent to the crank-pin circle.

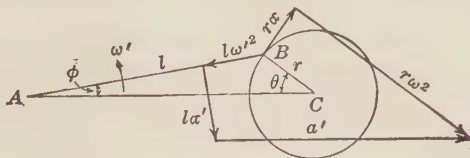


FIG. 260

347. In case the crank pin has an angular acceleration  $\alpha$ , show that the quadrilateral of Fig. 259 is replaced by the pentagon of Fig. 260, where  $\omega$  and  $\omega'$

must also satisfy the relation of Fig. 257. Hence show that if  $l$ ,  $r$ , either  $\omega$  or  $\omega'$ , and one of the quantities  $\alpha$ ,  $\alpha'$ ,  $a'$  are known for a given position of the connecting rod, the remaining three of the quantities may be determined by graphical construction. Follow through the details of the construction in each case to see that there is a definite solution.

348. From the pentagon in Fig. 260 show that

$$\begin{aligned} a' \cos \phi &= l\omega'^2 + r\omega^2 \cos(\theta + \phi) + r\alpha \sin(\theta + \phi), \\ l\alpha' \cos \phi &= r\omega^2 \sin \theta - r\alpha \cos \theta - l\omega'^2 \sin \phi. \end{aligned}$$

349. Given  $l = 6$  ft.,  $r = 1$  ft.,  $\omega = 2\pi$  rad./sec., and  $\alpha = \pi/4$  rad./sec.<sup>2</sup>, find the values of  $\omega'$ ,  $\alpha'$  and  $a'$  when  $\theta = 120^\circ$ . Make the graphical construction, and check by the formulas of the preceding problem.

350. Given  $l = 4$  ft.,  $r = 1$  ft.,  $\omega = 2\pi$  rad./sec.,  $a' = 150$  ft./sec.<sup>2</sup>,  $\theta = 30^\circ$ , find the values of  $\omega'$ ,  $\alpha'$ , and  $\alpha$  by graphical construction, and also by computation, using the formulas of Problem 348.

**119. Simultaneous Rotation of a Body about Two Intersecting Axes.** Consider a body mounted upon an axis  $OA$  (Fig. 261) about which it has pure rotation at a given instant; that is, there

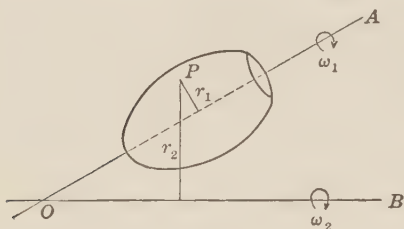


FIG. 261

is no motion of the particles of the body parallel to the axis at the instant in question. Let the angular velocity of the body about this axis be  $\omega_1$ . Suppose the axis  $OA$  to be rotating at the same instant about the line  $OB$  with angular velocity  $\omega_2$ .

This means that at the instant in question, any line of the body perpendicular to  $OA$  is turning with angular velocity  $\omega_1$  with respect to the plane  $AOB$  and at the same time the plane  $AOB$  is turning with angular velocity  $\omega_2$  about the line  $OB$ .

Consider now any point  $P$  of the body which at the instant is in the plane  $AOB$ . Let its distances from  $OA$  and  $OB$  be  $r_1$  and  $r_2$ , respectively. Its velocity at the instant is composed of the components  $r_1\omega_1$  and  $r_2\omega_2$ , both perpendicular to the plane  $AOB$ . For the point as chosen in the figure and with the indicated directions of rotation, these components act in the same directions, so that if the velocity of  $P$  is indicated by  $v_p$ , we have

$$v_p = r_1\omega_1 + r_2\omega_2.$$

Suppose vectors  $OM = \omega_1$  and  $ON = \omega_2$  (Fig. 262) to be laid off on  $OA$  and  $OB$  respectively, the directions being the directions a right-handed screw would advance with the assigned rotations. Then we may interpret the quantity  $r_1\omega_1 + r_2\omega_2$  as the sum of the moments of these vectors with respect to  $P$ . But this sum, by § 17, is equal to the moment of the resultant of the two vectors with respect to  $P$ . Hence if  $OC$ , the resultant of  $OM$  and  $ON$ , is drawn, and its distance from  $P$  is  $r$ , we have

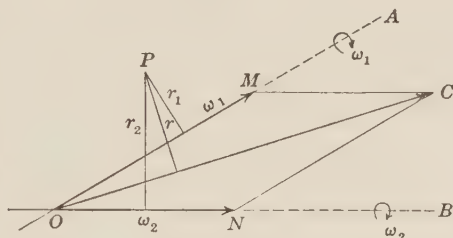


FIG. 262

$$r \cdot OC = r_1\omega_1 + r_2\omega_2, \quad \text{whence} \quad v_p = r \cdot OC.$$

If  $P$  is a point of the body on  $OC$ ,  $r = 0$ , and hence  $v_p = 0$ . Hence, every point of the body on  $OC$  has zero velocity at the given instant. Since the body is in motion, the motion at the given instant can only be one of rotation about  $OC$ , or  $OC$  is an *instantaneous axis of rotation* of the body.

Suppose the angular velocity of the body about  $OC$  at the given instant to be  $\omega$ . Then if  $P$  is any point of the body in the plane  $AOB$  distant  $r$  from  $OC$ , we have  $v_p = r\omega$ . But it was shown above that  $v_p = r \cdot OC$ . Therefore  $OC$  is equal to the magnitude of the angular velocity of the body about the instantaneous axis of rotation. Moreover,  $OC$  has the direction that a right-handed screw would advance if turned in the direction in which the body is rotating about  $OC$ .

Hence, if a body is rotating about an axis which at the same time is rotating about an intersecting axis, the motion of the body at any instant is equivalent to a rotation about an axis passing through the intersection of the two axes and lying in their plane. Moreover, the location of this instantaneous axis and the angular velocity of the body about it are obtained by laying off vectors on the axes equal respectively to the angular velocities about these axes. The resultant of these vectors lies



on the instantaneous axis, and its length is equal to the angular velocity of the body about the instantaneous axis.

Since the result is related to the two axes and the angular velocities of the body about them in precisely the same way, it is sufficient to regard the motion as one of simultaneous rotation of the body about the two axes.

We thus see that, with the conventions above made, angular velocity of a body may be regarded as a vector quantity, and is subject to the usual vector laws of resolution into components and of combining with other angular velocities into a resultant.

It therefore follows that if a body has pure rotation about a given axis at a given instant, its motion may be regarded as one

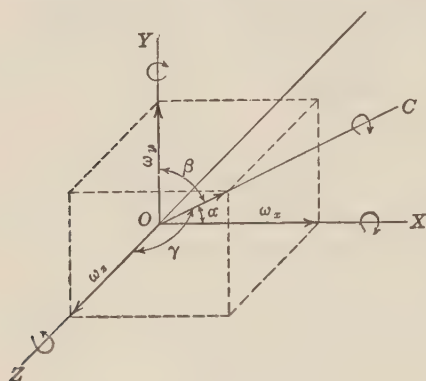


FIG. 263

of simultaneous rotation about three rectangular axes which intersect on the given axis, and the angular velocities about these axes are obtained by representing the angular velocity of the body as a vector quantity laid off on the axis of rotation and projecting this vector on the three rectangular axes. The projection on

each axis is a measure of the angular velocity of the body about that axis. Thus, in Fig. 263, if the body has angular velocity  $\omega$  about the line  $OC$  which makes angles  $\alpha$ ,  $\beta$ ,  $\gamma$  with axes  $OX$ ,  $OY$ ,  $OZ$  respectively, the motion may be regarded as equivalent to simultaneous rotations about these axes with angular velocity  $\omega_x$ ,  $\omega_y$ ,  $\omega_z$  respectively, where

$$\omega_x = \omega \cos \alpha, \quad \omega_y = \omega \cos \beta, \quad \omega_z = \omega \cos \gamma,$$

and

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2.$$

This representation is of great value in analyzing the motion of any rigid body, for example, the gyroscope.



**120. Foucault's Pendulum.** An interesting application of vector representation of angular velocity is seen in the use of Foucault's pendulum to show the rotation of the earth.

Let  $OP$  (Fig. 264) be the axis of the earth and let  $\omega$  be the angular velocity of the earth about its axis. Using the vector representation, the value of  $\omega$  is laid off on the axis of rotation. Considering any point  $A$ , with latitude  $\lambda$  on the earth's surface, the motion of the earth about its axis may be regarded as one of rotation about  $OA$  with angular velocity  $\omega \sin \lambda$ , at the same time that  $OA$  revolves about a perpendicular line  $OB$  with angular velocity  $\omega \cos \lambda$ . If a heavy weight attached to a fine wire is set swinging in a vertical plane through  $A$ , the plane of vibration will not turn about the line  $OA$ , but the rotation of the earth will cause an apparent rotation of the plane of the pendulum relative to the earth with angular velocity  $\omega \sin \lambda$ .

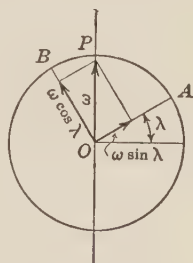


FIG. 264

Foucault, a French physicist, in 1851 thus demonstrated the rotation of the earth on its axis.

### PROBLEMS

351. A body is rotating at 30 r.p.m. about an axis  $OA$ , while at the same time the axis  $OA$  is making 40 r.p.m. about a line  $OB$  perpendicular to  $OA$ . Locate the instantaneous axis of rotation of the body at any instant and find the angular velocity of the body about this axis.

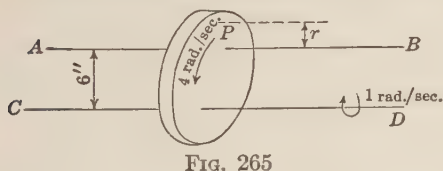


FIG. 265

352. Figure 265 represents a disk rotating about an axis  $AB$  with angular velocity 4 rad./sec., while at the same time the line  $AB$  is rotating about a parallel line  $CD$  distant

6" from  $AB$  with angular velocity 1 rad./sec. Without using the representation of angular velocity, show that if a point  $P$  of the disk in the plane of the two lines, distant  $r$  from  $AB$  and  $r + 6$  from  $CD$ , is chosen, the velocity,  $v_p$ , of  $P$  is

$$v_p = 4r - 1(r + 6).$$

Hence show that if  $r = 2''$ , the velocity of  $P$  is zero, and thereby locate the instantaneous axis of rotation.

Show also, by considering the velocity of a point of the disk on  $AB$ , that the angular velocity of the disk about the instantaneous axis is 3 rad./sec. and is in the direction of rotation of the disk about  $AB$ .

Prove that the same results are obtained by regarding the angular velocities as vector quantities.

353. In Fig. 265, suppose the angular velocity about  $CD$  to be 4 rad./sec. in the direction shown, all other conditions being as described in Problem 352. Show then that *all* points of the disk have velocity 24 in./sec. perpendicular to the plane containing  $AB$  and  $CD$ , and hence that the motion of the disk is one of pure translation.

Show that the same result is obtained by the use of vector representation of angular velocities.

354. A body is rotating about an axis with angular velocity  $\omega$  while the axis is rotating about a parallel line in the opposite direction with angular velocity numerically equal to  $\omega$ . If  $r$  is the distance between the axes, show that the body has a motion of pure translation and that the velocity of every point of the body is  $r\omega$ . (The side bar of a railroad engine, or the car of a Ferris wheel, is an example of this type of motion.)

355. Show that if a Foucault's pendulum be kept vibrating at a place of latitude  $\lambda$ , the time required for its plane to turn through  $360^\circ$  is  $24/\sin \lambda$  hrs. What does this become at the pole? At the equator? At your own location?

356. A body has angular velocity  $\omega_1$  about a line  $AB$  at the same time that  $AB$  is rotating about a parallel line  $CD$  in the same direction with angular velocity  $\omega_2$ . Show that the motion is equivalent to one of rotation with angular velocity  $\omega_1 + \omega_2$  about a line parallel to the lines  $AB$  and  $CD$ , lying in their plane and dividing the distance between  $AB$  and  $CD$  in the inverse ratio of the angular velocities. Prove this without the use of vector representation of angular velocities, and show that by the use of vectors the same result is obtained.

**121. Theorem of Coriolis. (Proof for Plane Motion).** Suppose a particle  $P$  to move along a plane path while at the same time the path has any motion in its own plane. We wish to find the acceleration of  $P$  when the motion of  $P$  in its path and the motion of the path are given.

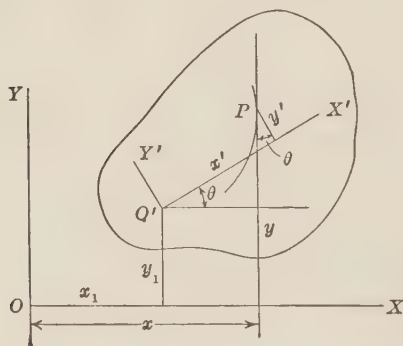


FIG. 266

Assume a portion of a plane surface containing the path to be rigidly attached to the path and moving with it, and let  $O'X'$  and  $O'Y'$  be rectangular axes fixed in this plane surface. Let  $OX$  and  $OY$  be fixed axes in space also in the plane in which the particle moves. Let the coordinates of  $P$  referred

to the fixed axes be  $(x, y)$  and referred to the moving axes be  $(x', y')$  and let  $\theta$  be the angle which  $O'X'$  makes with  $OX$ . Also let the coordinates of  $O'$  referred to the axes  $OX$  and  $OY$  be  $(x_1, y_1)$ . It is then evident from the figure (Fig. 266) that

$$x = x_1 + x' \cos \theta - y' \sin \theta,$$

$$y = y_1 + x' \sin \theta + y' \cos \theta.$$

The components of the acceleration of  $P$  parallel to the fixed axes are obtained by taking the second derivatives of  $x$  and  $y$  with respect to  $t$ . We thus find, letting  $\omega = d\theta/dt$ ,  $\alpha = d\omega/dt$ ,

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{d^2x_1}{dt^2} + \cos \theta \cdot \frac{d^2x'}{dt^2} - \sin \theta \cdot \frac{d^2y'}{dt^2} - 2\omega \sin \theta \frac{dx'}{dt} \\ &\quad - 2\omega \cos \theta \frac{dy'}{dt} - x' \omega^2 \cos \theta + y' \omega^2 \sin \theta - x' \alpha \sin \theta - y' \alpha \cos \theta, \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{d^2y_1}{dt^2} + \sin \theta \frac{d^2x'}{dt^2} + \cos \theta \frac{d^2y'}{dt^2} + 2\omega \cos \theta \frac{dx'}{dt} \\ &\quad - 2\omega \sin \theta \frac{dy'}{dt} - x' \omega^2 \sin \theta - y' \omega^2 \cos \theta + x' \alpha \cos \theta - y' \alpha \sin \theta. \end{aligned}$$

In these formulas  $d^2x_1/dt^2$  and  $d^2y_1/dt^2$  are the components of the acceleration of  $O'$  relative to the fixed axes, and  $\omega$  and  $\alpha$  are the angular velocity and angular acceleration of the moving axes relative to the fixed axes.

The formulas become much easier of interpretation if the axes are chosen in a particular way. Without loss of generality, we may now choose the fixed point  $O'$  of the moving plane to coincide with the position of  $P$  at the given instant, take the  $x'$  axis tangent to the path of  $P$  in the moving plane, with the positive end of the axis in the direction in which  $P$  is moving along the path, and take the fixed axis  $OX$  parallel to the moving axis  $O'X'$  at the instant in question. Then at the given instant,  $\theta = 0$ ,  $x' = 0$ ,  $y' = 0$ ,  $dy'/dt = 0$ ,  $dx'/dt = v$ , where  $v$  is the velocity of  $P$  in the moving plane. The equations for the components of acceleration then reduce to

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{d^2x_1}{dt^2} + \frac{d^2x'}{dt^2}, \\ \frac{d^2y}{dt^2} &= \frac{d^2y_1}{dt^2} + \frac{d^2y'}{dt^2} + 2\omega v. \end{aligned}$$

The first terms on the right of these equations are the components of the acceleration of the point in the moving plane which at the given instant coincides with the position of the particle, the second terms are the components of acceleration of the particle  $P$  with respect to the moving plane, and the term  $2\omega v$  is an additional acceleration perpendicular to the path in the fixed plane and directed toward the side of the tangent to the path toward which the vector  $v$  is turning due to the rotation of the moving plane containing the path of the particle.

The theorem may then be stated as follows. *If a particle moves along a path fixed in a plane surface while the plane surface moves in a fixed plane, the acceleration of the particle at any instant is made up of the following components:*

(1) *the acceleration of the particle relative to the path in the moving plane;*

(2) *the acceleration of the point of the path which at the given instant coincides with the position of the particle;*

(3) *a term  $2\omega v$  perpendicular to the path in the moving plane directed toward the side of the tangent to the path toward which the velocity vector relative to the path is rotating due to the rotation of the path, where  $v$  is the velocity of the particle relative to the moving plane and  $\omega$  is the angular velocity of the moving plane.*

This theorem is called the *Theorem of Coriolis*.

**122. Illustration.** As an illustration of the use of Coriolis' Theorem, consider a bar  $OA$  pivoted at a fixed point  $O$  and rotating about this point with constant angular velocity  $\omega$  in a counter-clockwise direction (Fig. 267). Pivoted at  $A$  is a second bar  $AP$  rotating about  $A$  in a counter-clockwise direction with constant angular velocity  $\omega'$  relative to  $OA$ .

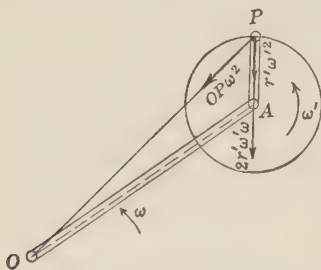


FIG. 267

If a portion of a plane be thought of as attached to  $OA$  and rotating with it, the path of  $P$  is a circle in this rotating plane. The plane is then turning with constant angular velocity  $\omega$  and the acceleration of the point of the path which at the given instant coincides with  $P$  is  $OP\omega^2$  directed toward  $O$ .

The velocity of  $P$  relative to the path is  $v = r'\omega'$  tangent to the circle, where  $r' = AP$ .

The acceleration of  $P$  relative to the path is  $r'\omega'^2$  directed toward  $A$ .

The supplementary term  $2\omega v$  is then  $2r'\omega'\omega$ , normal to  $v$  and directed toward  $A$ .

The vector sum of these component accelerations is the acceleration of  $P$ .

### PROBLEMS

357. In the illustration of § 122 let  $OA = 2$  ft.,  $AP = 6$  in.,  $\omega = 3$  rad./sec.,  $\omega' = 2$  rad./sec. Show that the acceleration of  $P$  when  $P$  is on  $OA$  between  $O$  and  $A$  is  $5.5$  ft./sec.<sup>2</sup> directed toward  $O$ , and when  $P$  is on  $OA$  on the opposite side of  $A$  from  $O$  the acceleration of  $P$  is  $30.5$  ft./sec.<sup>2</sup> directed toward  $O$ .

358. In the illustration of § 122, show that if either  $\omega$  or  $\omega'$  is reversed in direction, the vector  $2r'\omega'\omega$  is directed outward from the center of the circle.



## CHAPTER VII

### WORK AND ENERGY

**123. Definition of Work.** The simplest concept of work done by a force arises in the case of a constant force acting at a definite point of a body having a motion of straight line translation, where the force acts along the straight line in which its point of application moves. In this case, if the force is  $F$  and the distance moved by its point of application is  $s$ , the *work done by the force* in the displacement is defined to be  $Fs$ . If  $F$  is in pounds and  $s$  is in feet, the work is said to be in *foot-pounds*, abbreviated *ft.-lbs.* The unit of work is then the work done by a force of one pound acting through a distance of one foot. Other common units of work are the *gram-centimeter*, the work done by a force of one gram acting through a distance of one centimeter, and the *erg*, the work done by a force of one dyne acting through a distance of one centimeter.

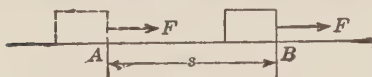


FIG. 268

Thus, in Fig. 268, if  $F$  is a constant force, in pounds, acting on a body during its displacement through a distance

$s$  feet in the direction in which  $F$  acts, the work done by  $F$  in the displacement is  $Fs$  ft.-lbs.

The work done by  $F$  in a given displacement is independent of the work that is done by any other forces that may be acting on the body.

If the displacement of the body is in the opposite direction to that in which the force acts, work is said to be done *against* the force, or the force is said to do *negative* work. Thus, in Fig. 268, if the block had moved from  $B$  to  $A$ , the work done by  $F$  would have been  $-Fs$  ft.-lbs.

**124. A More General Definition of Work of a Force.** The definition of work done by a force in the preceding article is of very limited application. It is necessary to extend the definition



to include more general conditions of force and motion. Of course the broader definition must contain the definition already given as a special case.

Consider now any force acting at any point of a moving body, Fig. 269. At a given instant let a force  $F$  act on the body at a point  $P$  and let the point  $P$  have a

velocity  $v$  at the instant. The product of the velocity  $v$  and the component of  $F$  in the direction of  $v$  is defined to be the **rate of work**, or the work per unit of time, of the force  $F$  at the given instant. Thus, if  $\phi$  is the angle measured from the vector  $v$  to the vector  $F$ , both with initial

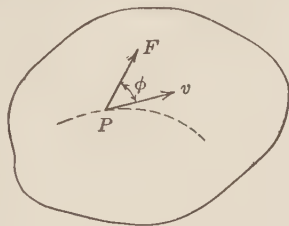


FIG. 269

points at  $P$ , the rate of work of  $F$  at the instant is  $Fv \cos \phi$ . The work done by  $F$  in a small interval of time  $dt$  is then  $Fv \cos \phi dt$ , approximately,<sup>1</sup> and the work done by  $F$  in the interval of time from  $t_1$  to  $t_2$  is

$$\int_{t_1}^{t_2} Fv \cos \phi dt.$$

In this definition it must be noted that at any instant  $v$  is the velocity of the *particle of the body* on which  $F$  is then acting. This particle may, or may not, be the same particle throughout the motion.

### 125. Illustrations of Work Done by a Force.

EXAMPLE 1. In Fig. 268, suppose that the force is constant and equal to  $F$  lbs. and that the point of application of the force remains the same throughout a displacement of the body through a distance of  $s$  ft. in the direction in which the force acts. If the time required for this displacement is  $t$  secs., then, by the definition of § 124, since here  $\phi = 0$ , the work done by  $F$  in the displacement is  $\int_0^t Fv dt$ . But at each instant  $v = ds/dt$  and hence the work done by  $F$  becomes

$$\text{work} = \int_0^t F \frac{ds}{dt} dt = \int_0^s F ds = Fs,$$

which agrees with the definition of § 123 for this special case.

<sup>1</sup> That is,  $Fv \cos \phi dt$  differs from the exact value of the work done by  $F$  by an infinitesimal of second order.

EXAMPLE 2. Assume a shaft of radius  $r$  to be rotating with constant angular velocity  $\omega$  in a bearing and that a constant friction force  $F$  acts

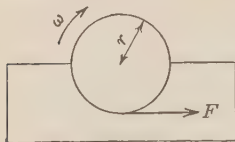


FIG. 270

on the shaft at its lowest point in the bearing, Fig. 270. Here the vector representing the force remains stationary but the point of application of the force at any instant has a velocity  $v = r\omega$ , and is opposite to the direction of the force. The time required for the shaft to make one revolution is  $2\pi/\omega$

and the work done *against*  $F$  in one revolution is therefore

$$\text{Work per revolution} = \int_0^{2\pi/\omega} F r \omega \, dt = F 2\pi r.$$

EXAMPLE 3. In Fig. 271 one block is placed on top of another. The surfaces of the blocks in contact are not smooth. A constant force  $P$  applied to the lower block is of such value as to move both blocks in the direction in which  $P$  acts, but so that the upper block slips back relatively to the lower.

In this motion a constant friction force  $F$  is developed at the surfaces in contact, acting forward (in the direction of  $P$ ) on the upper block, and an equal and opposite friction force is developed on the lower block.



FIG. 271

Suppose the lower block to move forward a distance  $s$  in  $t$  secs., and the upper block to move forward a distance  $s'$  in the same time, and let the velocities of the lower and upper blocks be  $v$  and  $v'$  at any instant of the motion. Then  $v = ds/dt$  and  $v' = ds'/dt$ .

We then have for the work done *against*  $F$  on the lower block (using the symbol  $U$  to represent work),

$$U_1 = \int_0^s F \frac{ds}{dt} dt = \int_0^s F \, ds = Fs.$$

Also the work done *by*  $F$  acting on the upper block is

$$U_2 = \int F \frac{ds'}{dt} dt = \int F \, ds' = Fs'.$$

Hence the net amount of work done against friction between the surfaces is  $F(s - s')$ , or, is equal to the product of the friction force between the surfaces and the distance through which one body slides relatively to the other.

**126. Work Done against Gravity in Raising a Weight.** First consider a particle of weight  $W$  which is moved along any path from a position  $A$  to a position  $B$  at a height  $h$  above  $A$  (Fig. 272). The work,  $U_g$ , done against the weight in the displacement is, by definition,

$$U_g = \int_0^t Wv \cos \phi \, dt$$

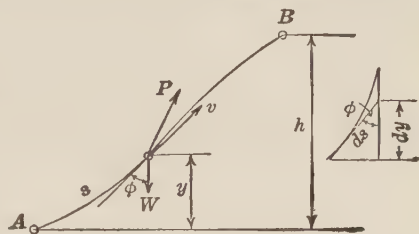


FIG. 272

where  $t$  is the time required for the displacement under the action of the given forces. But  $v = ds/dt$  and  $\cos \phi = dy/ds$ , and hence

$$U_g = \int_0^h W \, dy = Wh.$$

Next consider any body of weight  $W$  moved from one position to any other position (Fig. 273). The work done against the

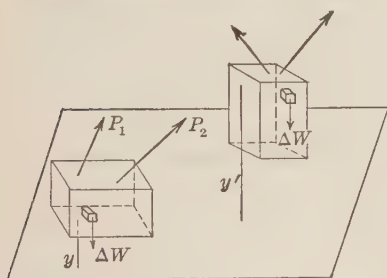


FIG. 273

weight of the body is the sum of the quantities of work done against the weights of the separate particles of which the body is made up. If  $\Delta W$  is the weight of a small element of the body,  $y$  and  $y'$  the heights of this element above a chosen horizontal plane in the initial and final

positions respectively of the body, the work done against the weight of the particle, by the above proof, is equal to  $\Delta W(y' - y)$ . Hence the total work done against the weight of the body in the displacement is

$$U_g = \int (y' - y) dW = \int y' dW - \int y dW = W(\bar{y}' - \bar{y})$$

where  $\bar{y}$  and  $\bar{y}'$  are the heights of the centroid of the body above the chosen reference plane in the initial and final positions of the body.

Hence, *the work done against the weight of a body when it is moved from one position to another is equal to the product of the weight of the body and the vertical distance through which the centroid of the body is raised.*

In the above discussion it is assumed that the change in position of the body is not sufficient to make an appreciable change in the earth's attraction for the body, that is the weight of the body.

**127. Work of a Force Expressed in Terms of Its Components.**  
In the formula for the work done by a force,

$$U = \int Fv \cos \phi \, dt,$$

or, since  $v = ds/dt$ ,

$$U = \int F \cos \phi \, ds,$$

the quantity  $F \cos \phi$  is the projection of  $F$  upon  $v$ , the velocity of the point of the body at which the force is applied and  $ds$  is the differential of arc of the curve along which this point is moving. If the force  $F$  has components  $X$ ,  $Y$ , and  $Z$  parallel to a set of fixed rectangular axes, and if  $v$  makes with these axes angles  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively, the projection of  $F$  upon  $v$  is equal to the sum of the projections of the components of  $F$  upon  $v$ , or

$$F \cos \phi = X \cos \alpha + Y \cos \beta + Z \cos \gamma,$$

and the work formula becomes

$$U = \int (X \cos \alpha \, ds + Y \cos \beta \, ds + Z \cos \gamma \, ds).$$

If  $dx$ ,  $dy$ , and  $dz$  are the differentials of the coordinates of the point of application of  $F$  when this point moves through the differential arc  $ds$ , then, since  $v$  has the same direction as  $ds$ ,

$$dx = ds \cos \alpha, \quad dy = ds \cos \beta, \quad dz = ds \cos \gamma,$$

and the work formula further reduces to the form

$$U = \int (X \, dx + Y \, dy + Z \, dz).$$

In the special case where  $F$  has always the same direction, this direction may be chosen as that of the  $x$  axis, and then  $X = F$ ,  $Y = 0$ ,  $Z = 0$ . The formula for the work then becomes

$$U = \int F dx.$$

In this formula it must be remembered that  $dx$  is the differential displacement of the point of the body to which the force is applied in the direction in which the force is acting. For example, the work done by the force of gravity on the particle of weight  $W$  when the particle is moved from  $A$  to  $B$  (Fig. 274) is

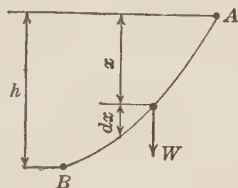


FIG. 274

$$U = \int_0^h W dx = Wh.$$

**128. Relation between Work Done on a Particle and Its Change in Velocity.** Consider a particle of mass  $M$  moved from

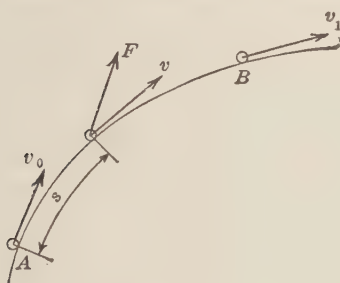


FIG. 275

a position  $A$  to a position  $B$  along any path (Fig. 275). At any instant let  $F$  be the resultant of all forces acting on the particle,  $\phi$  the angle which  $F$  makes with the velocity vector of the particle at that instant, and  $v_0$  and  $v_1$  the velocities of the particle at  $A$  and  $B$  respectively. The work done by all the forces acting on the

particle when it is displaced from  $A$  to  $B$  is then

$$U = \int Fv \cos \phi dt.$$

But  $F \cos \phi$  is the tangential component of all forces acting on the particle, and hence  $F \cos \phi = Mdv/dt$ . (See § 101). Therefore

$$U = \int_0^t M \frac{dv}{dt} v dt = \int_{v_0}^{v_1} Mv dv = \frac{1}{2} M(v_1^2 - v_0^2).$$

The same result evidently applies to any body having pure translation.

**129. Relation between Torque and Angular Acceleration of a Body Rotating about a Fixed Axis.** Assume a body to be rotating about a fixed axis under the action of a set of external forces. These forces we shall call *impressed forces*. At any instant let

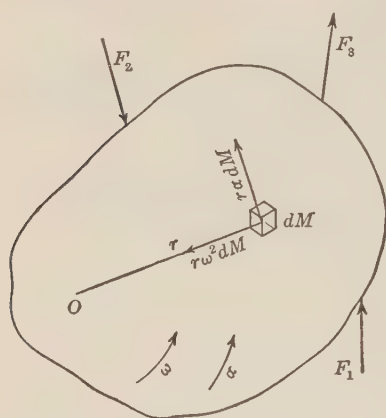


FIG. 276

the body have an angular velocity  $\omega$  and an angular acceleration  $\alpha$ . In Fig. 276 is represented a slice of the body contained between two parallel planes perpendicular to the axis about which the body is rotating. The axis cuts the face of this slice at O. Any element of the body of mass  $dM$ , distant  $r$  from the axis of rotation, has components of acceleration along the tangent and along the radius

of the circle in which it is moving of  $r\alpha$  and  $r\omega^2$ , respectively. Since we have assumed that the body is rotating about a fixed line through O, these accelerations are the only accelerations. The resultant of all forces acting on the particle can then be resolved into two components,  $r\alpha dM$  and  $r\omega^2 dM$ , perpendicular to  $r$  and along  $r$  as indicated in Fig. 276. These forces are called the *effective forces* acting on the particle. Evidently if the particle were detached from the body and were acted on at each instant by these forces it would have precisely the same motion that it has in moving with the body.

The effective forces acting on any particle of the body are, of course, due to any of the body's impressed forces that may act on that particle and to the forces that other particles of the body exert on it. These latter forces we shall call the *internal reactions* of the particles on each other. We may assume the internal reactions of the particles to occur in pairs, equal and opposite and in the same line, according to Newton's third law. If then there be formed any sum involving all of the effective forces acting on *all* of the particles of the body, it will be equal to the



similar sum of all of the impressed forces acting on the body and the similar sum of the internal reactions on all the particles of the body. Since these latter forces occur in pairs, equal and opposite, and in the same line, this sum is zero, and we may say that any sum of all of the impressed forces acting on the body is equal to the similar sum of the effective forces acting on all of the particles of the body. This statement is known as *D'Alembert's Principle*.

As an application of this principle we may say of the body under discussion that the sum of the moments of the impressed forces with respect to the axis of rotation is equal to the sum of the moments of the effective forces acting on all of the particles of the body with respect to the same axis. Thus we may write,

$$L = \int r \alpha dM \cdot r = \alpha \int r^2 dM = I\alpha,$$

where  $I$  is the moment of inertia of the body with respect to the axis of rotation and  $L$  is the sum of the moments, or *torque*, of the impressed forces with respect to that axis.

**130. Relation between the Work Done on a Rotating Body and the Change in Its Angular Velocity.** Suppose a body to have a motion of pure rotation about a fixed axis under the action of any set of impressed forces. Let each of the impressed forces be resolved into two components, one parallel to the axis of rotation and the other in a plane perpendicular to the axis of rotation and passing through the point of application of the force. Let  $F$  be the latter component of any impressed force and  $P$  its point of application at any instant (Fig. 277). The point  $P$  is moving in a circle of radius  $OP = r$ , where  $O$  is the point on the axis of rotation in the plane of  $F$ . Then when the body turns through an angle  $d\theta$  in the time  $dt$ , the work done by  $F$  is  $Fv \cos \phi dt$ .

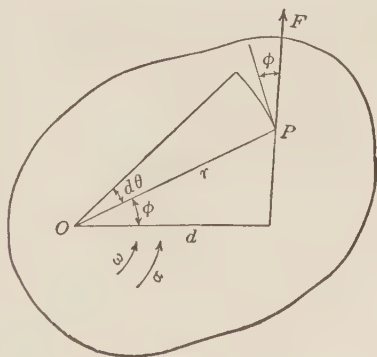


FIG. 277

But  $v = r d\theta/dt$  and  $\cos \phi = d/r$ , where  $d$  is the length of the perpendicular from  $O$  to the line of  $F$ , and hence the work done by  $F$  in the displacement  $d\theta$  becomes

$$F \frac{d}{r} r \frac{d\theta}{dt} dt = Fd d\theta.$$

There is no work done by any component of an impressed force parallel to the axis of rotation since at any instant the direction of the displacement of its point of application is perpendicular to the line of the force. Hence the work done by all of the impressed forces in the angular displacement  $d\theta$  is

$$dU = (\Sigma Fd)d\theta.$$

But  $\Sigma Fd$  is the torque of impressed forces with respect to the axis of rotation; and, by § 129, it is equal to  $I\alpha$ . Since  $\alpha = \omega d\omega/d\theta$ , this reduces to  $I\omega d\omega/d\theta$ , and hence we have

$$dU = I\omega d\omega.$$

Therefore the work done by all impressed forces in any angular displacement in which the angular velocity changes from  $\omega_0$  at the beginning of the period to  $\omega_1$  at the end, is

$$U = \int_{\omega_0}^{\omega_1} I\omega d\omega = \frac{1}{2} I(\omega_1^2 - \omega_0^2).$$



FIG. 278

**131. Illustrations. EXAMPLE 1. WORK OF COMPRESSING A SPRING.** Let there be given a spring, Fig. 278, which obeys Hooke's law: namely, that the compressive force is proportional to the deformation, or  $F = ks$ .

The work done in compressing the spring a distance  $s_1$  ft. is, by § 127,

$$U = \int_0^{s_1} ks \cdot ds = \frac{1}{2} ks_1^2,$$

where  $k$  is the force required to hold the spring compressed 1 ft.

**EXAMPLE 2.** A weight of 10 lbs. falls through a height of 5 ft. and strikes a coiled spring standing vertically, Fig. 279. If it requires a force of 100 lbs. to compress the spring 1 in., how much will the spring be compressed by the falling weight?

By § 128, the work done on the body in the displacement is equal to one-half its mass times the difference of the squares of its final and initial velocities. The weight starts from rest and at the instant of greatest compression of the spring the velocity of the weight is again zero. Hence the total work done on the weight in this motion is zero. The work done by the force of gravity is positive and the work done by the spring on the weight is negative, since the displacement is opposite to the direction of the force. If  $s$  is the number of feet the spring is compressed, we then have for the total work done on the weight,

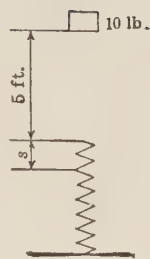


FIG. 279

$$(5 + s)10 - \int_0^s 1200s \cdot ds = 0,$$

or

$$50 + 10s - 600s^2 = 0.$$

Solving this quadratic equation, we find that the positive value of  $s$  which satisfies the equation is 0.297 ft., or 3.57 in.

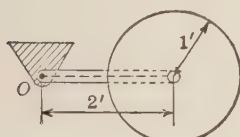


FIG. 280

EXAMPLE 3. A disk of radius 1 ft. is rigidly attached to a weightless bar in the plane of the disk, pivoted at a point  $O$ , 2 ft. from the center of the disk, Fig. 280. If the disk is released from rest when the center of the disk is at the same level as the pivot, what angular velocity will it have when it has turned through  $90^\circ$ ?

Here the forces acting on the body, composed of disk and bar, are the weight of the disk, acting at its center, and the reaction of the pivot at  $O$ . Since the velocity of the point of application of the latter force is zero throughout the motion, no work is done by the force. The work done by the force of gravity, by § 126, is equal to  $2W$ , where  $W$  is the weight of the disk. The work equation for the body is then, by § 130,

$$2W = \frac{1}{2}I\omega^2.$$

Here  $I$  is the moment of inertia of the body with respect to the axis of rotation; its value is

$$I = \frac{1}{2} \frac{W(1^2)}{32.2} + \frac{W(2^2)}{32.2} = \frac{4.5W}{32.2},$$

and hence

$$\omega^2 = 28.62;$$

therefore

$$\omega = 5.350 \text{ rad./sec.}$$

The velocity of the center of the disk is then

$$v = 2\omega = 10.70 \text{ ft./sec.}$$

If the bar had been pivoted to a smooth pin at the center of the disk, as indicated by the dotted line of Fig. 280, the disk would have descended without rotating and the formula of § 128 would apply. We should then have

$$2W = \frac{1}{2} \frac{Wv^2}{32.2},$$

from which  $v = 11.35 \text{ ft./sec.}$

EXAMPLE 4. Fig. 281 represents a brake wheel and drum mounted on a horizontal axle. A small rope wound around the drum (so that it cannot slip) is attached to a weight,  $W$ , of 200 lbs. The radius of gyration,  $k$ , of the rotor (the wheel and drum) with respect to the axis of rotation is 16 in. The weight of the rotor is 500 lbs. Other dimensions are as indicated in the figure. If no force is applied to the brake lever, and there is a constant friction force of 30 lbs. at the axle, what velocity will be acquired by the falling weight in a descent of 40 ft.? Show also that the acceleration of the weight and the tension in the rope are constant and find their values. After a descent of 40 ft., what constant force

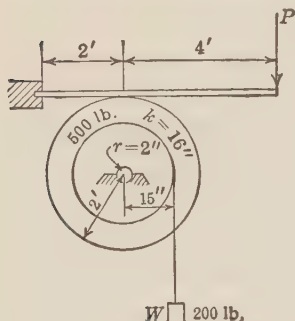


FIG. 281

$P$  must be applied at the end of the lever to bring the weight to rest in a further descent of 20 ft., if the brake friction force is one-fifth of the normal force between wheel and lever, and if the axle friction becomes 33 lbs. when the brake lever is applied?

Before the brake is applied, the forces acting on the rotor are its weight, the normal reaction of the bearing, the axle friction, and the tension  $T$  in the rope, which may be regarded as a force applied at the circumference of the drum at the point where the rope leaves the drum. The forces acting on  $W$  are its weight of 200 lbs. acting downward, and the tension  $T$  acting upward. The work equations for the weight and the rotor when the weight has descended a distance  $s$  ft. from rest are then by §§ 128 and 130, respectively, as follows:

$$(1) \quad 200s - \int_0^s T \, ds = \frac{1}{2} \frac{200}{32.2} v^2,$$

$$(2) \quad \int_0^s T \, ds - (30) \left( \frac{2}{15} s \right) = \frac{1}{2} \frac{500}{32.2} \left( \frac{16}{12} \right)^2 \omega^2,$$

By the addition of equations (1) and (2), we obtain

$$(3) \quad 196s = \frac{1}{2} \frac{200}{32.2} v^2 + \frac{1}{2} \frac{500}{32.2} \left( \frac{4}{3} \right)^2 \omega^2.$$

Since the rope is not slipping on the drum, the velocity of  $W$  is the same as the velocity of the drum at the point where the rope leaves the drum, or  $v = 15\omega/12$ . If  $\omega = 4v/5$  is substituted in (3), the resulting equation is

$$(4) \quad v^2 = 16.417s.$$

When  $s = 40$  ft.,  $v^2 = 656.7$ ,  $v = 25.63$  ft./sec.

It may be noted that the form of equation (4) shows that the acceleration of  $W$  is constant, for on taking the derivatives of both sides of the equation with respect to the time, we get

$$2v \frac{dv}{dt} = 16.417 \frac{ds}{dt},$$

or, since  $v = ds/dt$  and  $dv/dt = a$ , the acceleration of  $W$ , we find

$$a = 8.21 \text{ ft./sec.}^2$$

It follows then that the tension  $T$  is constant throughout this part of the motion. To obtain its value we have

$$200 - T = \frac{200}{32.2} (8.21),$$

from which  $T = 149.0$  lbs.

Assume that when the weight has descended 40 ft., a constant force  $P$  is applied at the end of the brake lever, of such value that the weight is brought to rest in a further descent of 20 ft. The normal force between brake and wheel then becomes  $3P$  and the brake friction  $3P/5$ , or  $0.6P$ . Using the assumed axle friction of 33 lbs. and assuming that the tension in the rope is  $T'$ , we find that the work equations for the weight and rotor after the weight descends a further distance of  $s$  ft., are respectively, since  $\omega = 4v/5$ ,

$$(5) \quad 200s - \int_0^s T' ds = \frac{1}{2} \frac{200}{32.2} (v^2 - 656.7),$$

$$(6) \quad \int_0^s T' ds - (33) \left( \frac{2}{15} s \right) - (0.6P) \left( \frac{24}{15} s \right) \\ = \frac{1}{2} \frac{500}{32.2} \left( \frac{4}{3} \right)^2 \left( \frac{4}{5} \right)^2 (v^2 - 656.7).$$

By the addition of equations (5) and (6), we obtain

$$(7) \quad 195.6s - 0.96Ps = 11.94(v^2 - 656.7).$$

If now  $v = 0$ , when  $s = 20$ ,

$$3912 - 19.2P = -7841,$$

and

$$P = 612.1 \text{ lbs.}$$

Substituting this value of  $P$  in (7), and taking the derivatives of both sides with respect to the time, we find the acceleration to be  $-16.42 \text{ ft./sec.}^2$ . The equation of motion of the weight then becomes

$$200 - T' = -\frac{200}{32.2}(16.42),$$

from which

$$T' = 302 \text{ lbs.}$$

### PROBLEMS

359. A body of weight 200 lbs. slides from rest down a plane inclined  $20^\circ$  to the horizontal. A constant friction force of 15 lbs. retards the body. By the use of the work equation find the velocity when the body has moved 60 ft.

360. By the use of the work equation, show that when a particle moves along any smooth path under the action of its own weight and the reaction of the path only, the relation between the velocities of the particle at two points of the path distant  $h$  apart vertically is given by  $v_1^2 - v_2^2 = 2gh$ , where  $v_1$  is the velocity of the particle in the lower position.

361. A freight car weighing 20 tons is moving along a horizontal track at the rate of 5 mi./hr. with brakes set when it strikes a bumping post. If the friction between the wheels and the brakes is 8,000 lbs., and if the post is equipped with a spring which requires a force of 60,000 lbs. to compress it one inch, how much will the spring be compressed by the car? How far back will the car be thrown? Neglect work required to stop rotation of wheels.

362. The spring of Fig. 282 requires a force of 500 lbs. to compress it one inch. It is compressed a distance of 4 in. and released with one end fixed and the other against a 5-lb. weight. If there is no friction, what velocity will the weight have when the spring reaches its natural length? If with this velocity the weight strikes another horizontal spring which requires a force of 1000 lbs. to compress it one inch, how much will the second spring be compressed?



FIG. 282

In this and the following problems the inertia of the spring and the internal friction are neglected. In general these effects are small, but if the mass of the spring is not small compared to that of the weight, the results may be considerably changed.



363. A weight  $W$  suspended from a spring extends the spring  $s_0$  in. when hanging at rest. If when the spring is hanging vertically and is of natural length, the weight is attached to the spring and is released from rest, show that the weight will descend  $2s_0$  in. before being retracted by the spring's elasticity. Show also that the weight has its maximum velocity when it has descended  $s_0$  in. How much work is done in stretching the spring the first  $s_0$  in., the second  $s_0$  in.?

364. A weight of 20 lbs. falls 10 ft. and strikes a coiled spring standing vertically. If it requires 50,000 lbs. to compress the spring one inch, how much will the weight compress the spring? What maximum force will the spring exert on the weight? Solve also if the spring requires only 25,000 lbs. to compress it one inch.

365. The wheel of Fig. 283 has a moment of inertia  $I$  about its axis of rotation. Assume that there is no axle friction. Show that when  $W$  descends a distance  $s$  ft. from rest, the velocity of  $W$  is given by

$$v^2 = \frac{2gr^2sW}{Wr^2 + Ig}.$$

Given  $W = 50$  lbs.,  $r = 2$  ft.,  $I = 38.82$ , show that when  $W$  descends 100 ft. from rest, the velocity attained is equivalent to that acquired in a free fall of 13.8 ft.

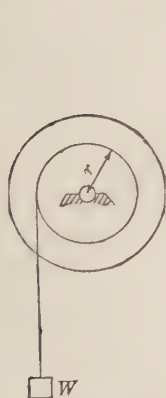


FIG. 283

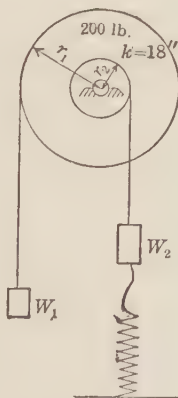


FIG. 284

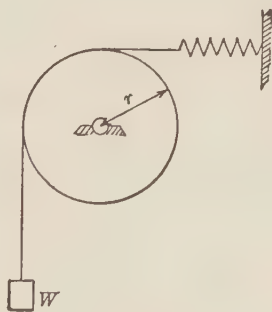


FIG. 285

366. In Fig. 284  $W_1 = 120$  lbs.,  $W_2 = 180$  lbs.,  $r_2 = 8$  in.,  $r_1 = 2$  ft. Assume that there is no axle friction. The system is released from rest when  $W_2$  is at the top of the uncompressed spring. An inextensible string 3 ft. long connects  $W_2$  and the top of the spring. If it requires 400 lbs. to stretch the spring one inch, how much will the spring be stretched by the weights before they are brought to rest?

367. In the apparatus of Fig. 285 the string does not slip on the wheel and axle friction is to be neglected. The wheel has a moment of inertia  $I$  about the axis of rotation. The spring is of such strength that a force of  $W$  lbs. applied to it stretches it  $s_0$  in. If  $W$  is released from rest when the spring is of natural

length and the string is tight, show that  $W$  will descend a distance of  $2s_0$  in., a result that does not depend upon the size nor upon the weight of the wheel. Show also that the work equation of the system, obtained by combining the separate work equations of the wheel and the weight, for a descent of  $W$  through  $s$  in., is

$$Ws - \frac{1}{2} \frac{Ws^2}{s_0} = 12 \left( \frac{1}{2} \frac{Wv^2}{32.2} + \frac{1}{2} \frac{Iv^2}{r^2} \right),$$

where  $r$  is in ft., and  $v$  is in ft./sec. Hence show that the maximum velocity of the weight occurs when  $s = s_0$ , and that the value of this maximum velocity is

$$\left\{ \frac{16.1Ws_0r^2}{6(Wr^2 + 32.2 \cdot I)} \right\}^{1/2} \text{ ft./sec.}$$

368. Show that the motion in the preceding problem is periodic and that the period of vibration is increased by increasing the value of  $I$ . Given  $W = 40$  lbs.,  $s_0 = 2$  in.,  $I = 4$ ,  $r = 2$  ft., prove that the period of vibration is 0.607 sec. Show also that if the surface of the wheel were frictionless, the period would be 0.452 sec.

369. A body is suspended by two strings each of length 4 ft., attached to supports so that when the weight hangs at rest the strings are parallel. If the body is pulled aside in the plane of the strings until the strings are inclined  $60^\circ$  to the vertical and released, show that the centroid of the body when it reaches its lowest position will have a velocity of 11.35 ft./sec.

**132. Work Done by Internal Forces on a System of Connected Bodies.** When a system of two or more bodies is in motion, there may occur pairs of equal and opposite forces acting on two of the bodies in such a way that the points of application of the forces have at each instant equal velocities, in both magnitude and direction. The work done on the system by one of the forces is then equal and opposite in sign to that done by the other force of the pair. In such a case the work equation of the whole system may be written, leaving out of account all such internal pairs of forces. Thus, in Example 4 of § 131, since the forces in the cord acting on the wheel and on the weight are equal and opposite in direction and their points of application have at each instant equal velocities, equation (3) might have been written as the work equation of the system of bodies rather than as the result of combining the work equations of the separate bodies.

**133. Work Done by Mutual Forces Acting between Two Particles.** Assume two particles to be acted upon by two equal and opposite forces,  $F$  and  $F'$ , acting in the line joining the

particles and in opposite directions (Fig. 286). By definition, the rate at which  $F$  is doing work on  $M_1$  is  $Fv \cos \phi$ , (§ 124), which may be interpreted as the product of  $F$  and the component of the velocity of  $M_1$  in the direction in which  $F$  acts. In like manner the rate at which  $F'$  is doing work on  $M_2$  is the product of  $F'$  and the component of the velocity of  $M_2$  in the direction in which  $F'$  acts. Since  $F$  and  $F'$  are numerically equal, and in the figure assumed to be attractive forces, that is, the force on each particle acts toward the other particle, the sum of the rates of work by  $F$  and  $F'$  is the product of the force of attraction and the

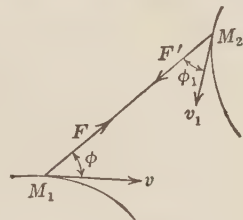


FIG. 286

rate at which the particles are approaching each other. If  $r$  is the distance between the two particles at the instant, the rate of approach of the particles is  $-dr/dt$ , and hence the rate of work by the two forces is  $-F dr/dt$ . The work done in the time  $dt$  is then  $-F dr$ , and the work done when  $r$  is changed from  $r_1$  to  $r_2$  is

$$U = - \int_{r_1}^{r_2} F dr.$$

If the forces are repulsive forces, the formula evidently is changed in sign only.

If the particles keep always at the same distance apart, the work done by the mutual forces is zero. For example, the work done by the internal reactions of any two elements of mass of a rigid body is zero in any displacement.

**134. Work Done by Impressed Forces on a Rigid Body Having Plane Motion.** Suppose a body having plane motion to be acted upon by any set of impressed forces. The value of the work done on the body in any displacement is evidently the sum of the quantities of work done on all of the particles of the body by all of the forces acting on these particles. Now the forces acting on the particles are composed of the impressed forces acting on the body and the internal reactions of the particles on each other. The work done by these latter forces, by § 133, is equal to zero, and hence the work done by the impressed forces is equal to the

sum of the quantities of work done on all of the particles of the body. But it was shown in § 128 that the work done on a particle of mass  $M$  when its velocity is changed from  $v_0$  to  $v_1$  is equal to  $\frac{1}{2}M(v_1^2 - v_0^2)$ . Therefore the total work done on the body by the impressed forces is

$$(1) \quad U = \frac{1}{2} \int v_1^2 dM - \frac{1}{2} \int v_0^2 \cdot dM,$$

where  $v_0$  and  $v_1$  are the initial and final velocities of  $dM$  and, in general, are different for different particles of the body.

In § 118, it was shown that any plane motion is equivalent to a rotation about an instantaneous axis at each instant. Let then Fig. 287 represent an elementary slice of the body contained between two adjacent planes perpendicular to the instantaneous

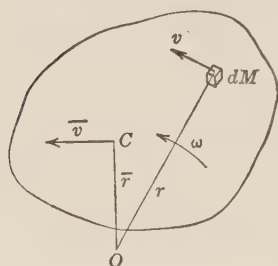


FIG. 287

axis, and let  $O$  be the point where one of these planes is cut by the instantaneous axis at a given instant. If the angular velocity of the body at the instant is  $\omega$ , the velocity of any element of mass  $dM$  of the body distant  $r$  from the instantaneous axis is  $v = r\omega$ , and therefore the value of  $\frac{1}{2} \int v^2 dM$  becomes  $\frac{1}{2} \int r^2 \omega^2 dM$  or  $\frac{1}{2} I_0 \omega^2$ , where  $I_0$  is the moment of inertia of the body with respect to the

instantaneous axis at the instant. Since the instantaneous axis changes with the time, it is more convenient to use another axis of reference. If the centroid of the body is distant  $\bar{r}$  from the instantaneous axis and  $\bar{v}$  is the velocity of the centroid, then  $\bar{v} = \bar{r}\omega$ . Also if  $\bar{I}$  is the moment of inertia of the body with respect to a line through the centroid parallel to the instantaneous axis,  $I_0 = \bar{I} + M\bar{r}^2$ . If we substitute this value in  $\frac{1}{2} I_0 \omega^2$ , we reduce it to the form  $\frac{1}{2} \bar{I} \omega^2 + \frac{1}{2} M\bar{r}^2 \omega^2$ , or  $\frac{1}{2} M\bar{v}^2 + \frac{1}{2} \bar{I} \omega^2$ . The work equation (1), then becomes

$$U = \frac{1}{2} M(\bar{v}_1^2 - \bar{v}_0^2) + \frac{1}{2} \bar{I}(\omega_1^2 - \omega_0^2).$$

**135. Kinetic Energy of a Body.** The work that must be done upon a body by impressed forces to bring it from a state of rest to a state of motion is called the *kinetic energy* of the body in

that state of motion. Thus the work required to bring a body of mass  $M$  from a state of rest to a state of motion of translation with velocity  $v$  has been shown to be  $\frac{1}{2}Mv^2$  (§ 128). The quantity  $\frac{1}{2}Mv^2$  is then a measure of the kinetic energy of a body of mass  $M$  having a motion of translation with velocity  $v$ . It is clear also from § 128 that if the body has velocity  $v$  and if work is done against it to bring it to rest, the measure of this work is again  $\frac{1}{2}Mv^2$ . Hence the kinetic energy of a body in translation may be taken as a measure of the work that must be done upon the body to reduce it to rest, or, as the work that the body can do in giving up its velocity, or, again, as a measure of the work stored in the body due to its velocity.

In similar manner, the kinetic energy of a body in pure rotation is  $\frac{1}{2}I\omega^2$ , (§ 130), and the kinetic energy of a body having any plane motion is  $\frac{1}{2}M\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$ , (§ 134).

The unit of kinetic energy is, of course, the same as the unit of work. Thus, a body of weight 20 lbs., in translation with a velocity 15 ft./sec., has  $\frac{1}{2}(20)(15)^2/32.2$  ft.-lbs. of kinetic energy; a body of weight 100 lbs., rotating about a fixed axis with angular velocity 5 rad./sec. and having a radius of gyration with respect to the axis of rotation of 2 ft., has  $\frac{1}{2}(100)(2^2)(5^2)/32.2$  ft.-lbs. of kinetic energy.

Having in mind the above definition of kinetic energy, we may interpret the meaning of the three equations

$$(1) \quad U = \frac{1}{2}M(v_1^2 - v_0^2),$$

$$(2) \quad U = \frac{1}{2}I(\omega_1^2 - \omega_0^2),$$

$$(3) \quad U = \frac{1}{2}M(\bar{v}_1^2 - \bar{v}_0^2) + \frac{1}{2}\bar{I}(\omega_1^2 - \omega_0^2),$$

as follows:

*The work done upon a body in any displacement (involving plane motion) is equal to the gain in kinetic energy in that displacement.*

By "gain in kinetic energy in a displacement" is meant the kinetic energy at the end of the displacement minus the kinetic energy at the beginning of the displacement. In computing the work done upon the body by any force, it must be remembered that the work done by the force is positive or negative according



as the component of the force along the line of the velocity of its point of application has the same or the opposite direction as the velocity vector of that point.

### PROBLEMS

370. A uniform sphere and a uniform disk of equal masses are rolling without slipping on a level track. If the velocities of their centers are equal, show that the kinetic energy of the sphere is  $14/15$  of the kinetic energy of the disk.

371. The total weight of a car and its load is 40 tons. There are four pairs of wheels. Each pair of wheels and axle weighs 1600 lbs. and has a radius of gyration of 0.85 ft. with respect to the central axis. What per cent of the total energy of the car is rotational? When moving on a level track at the rate of 5 mi./hr. the brakes are applied, and they develop a frictional force of 8000 lbs. between brake and wheels. If there is an additional backward force of 300 lbs. due to the track resistance and axle friction, how much farther will the car move before coming to rest? The wheel diameters are 33 in.

372. A uniform sphere and a uniform disk are started from rest at the same instant from points on a plane inclined at an angle  $\theta$  to the horizontal, and they roll down the plane without slipping. Show that when they have rolled a distance  $s$ , the velocities are given by the following formulas: for the sphere,  $v^2 = (10/7)g \sin \theta \cdot s$ ; for the disk,  $v^2 = (4/3)g \sin \theta \cdot s$ . Hence show that the center of each moves with uniform acceleration and that in a given time the distance the disk rolls is  $14/15$  of the distance the sphere rolls.

(SUGGESTION. Use the work equation, noting that the normal and frictional forces exerted by the plane on the rolling body do no work, since the point of the body at which they act has zero velocity at each instant, and hence that the weight is the only impressed force that does work on the body.)

373. A uniform disk and a hollow sphere of uniform density, and with inner radius equal to half the outer, are started from rest at the same instant and roll down an inclined plane without slipping. Prove that in a given time the disk will roll  $101/105$  of the distance rolled by the hollow sphere.

374. The wheel  $A$  of Fig. 288 has a radius of 10 in. and a radius of gyration with respect to a central axis of 8 in. Its weight is 120 lbs. The block  $B$  weighs 200 lbs. The wheel rolls without slipping and the block is acted on by a frictional force of 40 lbs. Find the velocity of  $B$  when it has travelled 20 ft. from rest. Show that the acceleration of  $B$  is constant, and find the force in the bar connecting  $B$  and  $A$ . Solve also on the supposition that  $B$  is on small frictionless rollers.

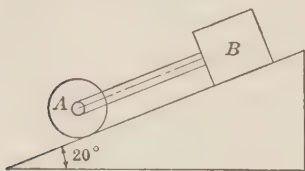


FIG. 288

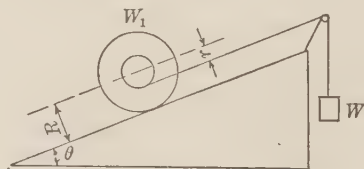


FIG. 289



375. The thin flexible cord of Fig. 289 is wrapped around and is attached to the axle of radius  $r$ , which is rigidly connected to the wheel of radius  $R$ . Given that there is sufficient friction between the wheel and inclined plane to prevent the wheel from slipping, prove that if the connected weights are released from rest the wheel will roll up the plane, remain at rest, or roll down the plane, according as  $W(R - r)$  is greater than, equal to, or less than  $W_1 R \sin \theta$ . Show that in any case when the weight  $W$  moves a distance  $s$ , the center of the wheel moves a distance  $Rs/(R - r)$ .

If  $W = 50$  lbs.,  $W_1 = 60$  lbs.,  $R = 2$  ft.,  $r = 8$  in., radius of gyration of wheel = 18 in.,  $\theta = 30^\circ$ , find the velocity of  $W$ , the velocity of the center of the wheel, and the angular velocity of the wheel when  $W$  has moved 20 ft. from rest. Show that the tension in the cord is constant throughout the motion; and find its value.

376. If the force with which the earth attracts another (relatively small) body is  $k/r^2$ , where  $r$  is the distance from the center of the earth to the body, show that the work done by the force of attraction on the body when it is brought from a great distance to the surface of the earth is nearly  $k/R$  units of work, where  $R$  is the radius of the earth. Show also that the value of  $k$  is  $WR^2$ , where  $W$  is the weight of the body at the earth's surface. Then show that if the velocity of the body at a great distance were small, the velocity it would have on reaching the earth's surface would be, except for air resistance, nearly 7 mi./sec.

**136. Work Done by a Couple Acting on a Body in Plane Motion.** Suppose a body to have plane motion and let Fig. 290 represent a section of the body in the plane of motion. Let  $F$  and  $F'$  be two forces forming a couple in this plane and acting on the body. If  $O$  is the point where the instantaneous axis of rotation cuts the plane of motion, and if  $\omega$  is the angular velocity of the body about the axis through  $O$  at the instant, the work done by  $F$  when the body turns through an angle  $d\theta$  is  $F(d + r)d\theta$ , and that done by  $F'$  is  $-F'r d\theta$  (§ 130). The work done by the two forces forming the couple is then  $(Fd)d\theta$ . Hence when the body turns through an angle  $\theta$ , the work done by the couple is

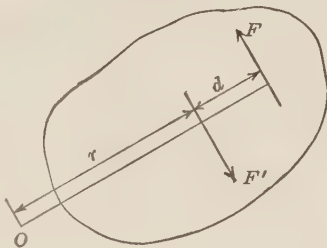


FIG. 290

$$U = \int_0^\theta (Fd)d\theta.$$

If the moment of the couple is constant, the formula becomes

$$U = (Fd)\theta,$$

or the product of the moment of the couple and the angle turned through by the body in the direction of rotation of the couple.

If the couple is not in the plane of motion of the body, it can be resolved into two component couples, one in the plane of motion and the other in a plane perpendicular to the plane of motion. The forces of the latter couple do no work and the work done by the first couple is given by the formula just derived.

**137. Power.** *Power* is the rate of doing work. In English-speaking countries a common unit of power is the *horsepower*, abbreviated *hp.*, which is 550 ft.-lbs. per sec. Thus, a motor which does 1100 ft.-lbs. of work per sec. is said to develop 2 horsepower. In the centimeter-gram-second system, power may be reckoned in *dyne-centimeters*, or *ergs, per second*. A more commonly used unit is the *watt*, which is  $10^7$  ergs per second. The *kilowatt* is  $10^{10}$  ergs per second. The *metric horsepower*, which is 75 kilogram-meters per second, is also used. One horsepower is equivalent to 746 watts and the continental horsepower to 736 watts.

### PROBLEMS

377. A car of weight 2500 lbs. is moving on a level road against a road resistance of 100 lbs. If when the speed is 10 mi./hr., the acceleration of the car is 6 ft./sec., what hp. is the engine of the car developing? What hp. if the speed is 40 mi./hr. and the acceleration 3 ft./sec.?

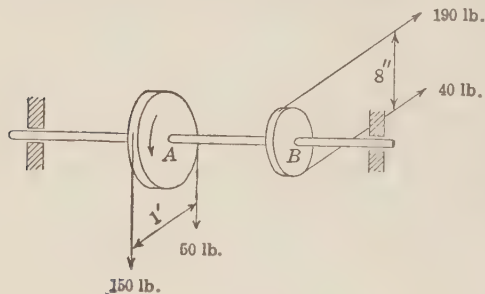


FIG. 291

378. The tensions in the belt around pulley A (Fig. 291) are 150 lbs. and 50 lbs. and at pulley B the tensions are 190 lbs. and 40 lbs. as shown in the figure. When the shaft is making 180 r.p.m. in the direction indicated, show that it is transmitting 1.71 hp. from pulley A to pulley B.

379. Steam from a boiler enters the cylinder of an engine under constant pressure until the piston moves a distance  $a$  from one end. At this point the

steam is cut off and the steam then in the cylinder expands according to the law,  $\text{pressure} \times \text{volume} = \text{constant}$ , to the end of the stroke. If  $P$  is the total pressure before cut-off,  $a$  the distance the piston moves before cut-off,  $h$  the length of stroke, and  $R$  the total constant back-pressure of the steam being expelled by the piston, show that the work done by the steam per stroke is

$$Pa + Pa \int_a^h \frac{ds}{s} - Rh.$$

380. Steam enters an engine cylinder under a constant pressure of 200 lbs. per sq. in. The steam is cut off when the piston has moved 8 in. from the end. There is a constant back-pressure of 10 lbs. per sq. in. during the exhaust. The length of stroke is 24 inches. The area of the piston is 75 square inches. The engine is double acting. What horsepower is being generated by the steam when the engine is making 80 r.p.m.? Assume the same law of expansion as in Problem 379. (Horsepower computed in this way is called the *indicated horsepower*. No account is taken of frictional losses.)

381. Fig. 292 represents a steam hammer. If  $W$  is the weight of the moving parts,  $h$  the length of stroke,  $P$  the total constant pressure before cut-off,  $R$  the total constant back-pressure, and if after cut-off the law of the expanding steam is  $\text{pressure} \times \text{volume} = \text{constant}$ , show that the velocity  $v$  of the hammer at the end of the stroke is given by

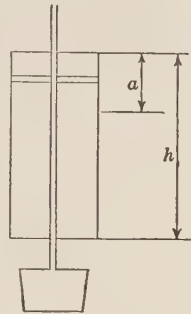


FIG. 292

$$\frac{1}{2} \frac{W}{g} v^2 = Wh + Pa \left[ 1 + \log_e \left( \frac{h}{a} \right) \right] - Rh.$$

Show also that the hp. delivered by the engine operating the hammer must average at least  $NWv^2/(66\,000g)$ , where  $N$  is the number of strokes per minute made by the hammer.

**138. Friction; Coefficient of Friction.** In preceding portions of this book occasional reference has been made to frictional forces, but no exact description of such forces has been given. We shall now discuss the subject of friction in a little more detail, but only an elementary treatment of some of the simpler cases will be undertaken.

When the plane surface of one body moves, or tends to move, along the plane surface of another body with which it is in direct contact, there is a resistance to the motion along the surface, which is known as *friction*. If the bodies are at rest relative to each other, the force is known as *static friction*, and if they are in motion with respect to each other the friction is called *kinetic friction*. Each body at the same time exerts on

the other a force perpendicular to the common surface, known as the *normal force*. The normal force and the friction force that act on one of the bodies are numerically equal but are opposite in direction to the corresponding forces that act on the other body. The resultant of the normal force and the friction force that act on one of the bodies is called the *reaction* of the other body upon it, or, more simply, the *reaction between the bodies*.

If the surfaces of the bodies in contact are not plane surfaces, the preceding definitions are applicable to any infinitesimal element of the surfaces in contact.

As illustrations of these definitions consider the bodies represented in Figs. 293 and 294. In Fig. 293 a force  $P$  is applied to a body in contact with a horizontal plane surface which offers a

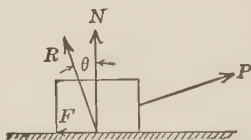


FIG. 293

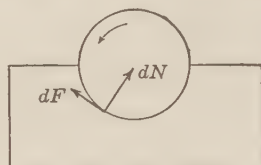


FIG. 294

resistance to the motion of the body. This resisting force, represented by  $F$ , is the *friction* acting on the block, and the force  $N$ , exerted by the plane on the block at right angles to the common surface, is the *normal force* on the block. The resultant,  $R$ , of  $F$  and  $N$  is the *reaction* of the surface on the block. The frictional force  $F$  may be regarded as the resultant of all of the frictional forces acting on all of the elementary surfaces of the block in contact with the plane. Again, in Fig. 294,  $dF$  and  $dN$  represent the frictional force and the normal force acting on an element of the surface of the shaft in contact with the surface of the bearing as the shaft turns, or tends to turn, in the direction indicated. We again define the total friction acting on the shaft as the sum of the frictional forces on the elements of area, but this sum is the arithmetic sum and not the vector sum, since all the forces must act to oppose the rotation of the shaft on its axis.

When the bodies are at rest relatively to each other but are just on the point of relative motion along the common surface,

the friction is called the *limiting friction*. If the bodies are not on the point of relative motion, the friction is less than the limiting friction and changes as the other forces applied to the bodies are changed.

As the results of experiment the following approximate law of friction may be stated. *The limiting friction between two bodies is directly proportional to the normal force acting between the bodies.*

The ratio of the limiting friction to the normal force is called the coefficient of static friction, or, briefly, the *coefficient of friction*. Thus, denoting the coefficient of friction by  $f$ , and assuming the friction to be limiting friction, we have, in Fig. 293,

$$f = \frac{F}{N},$$

and in Fig. 294,

$$f = \frac{dF}{dN}.$$

The angle which the resultant of the limiting friction and the normal force makes with the normal force is called the *angle of friction*. Thus, in Fig. 293, if  $F$  is the limiting friction,  $N$  the normal force, and  $R$  the resultant of  $F$  and  $N$ , then the angle  $\theta$  between  $N$  and  $R$  is the angle of friction.

It follows from the definition that  $f = \tan \theta$ , or, the coefficient of friction is equal to the tangent of the angle of friction.

The coefficient of friction depends upon the nature and condition of the surfaces in contact. Thus, on unlubricated surfaces, for wood on wood, the coefficient of friction usually ranges between 0.25 and 0.50; for metal on wood, between 0.20 and 0.60; and for metal on metal, between 0.15 and 0.30. By proper lubrication the coefficient of friction may be greatly reduced. In this case the coefficient of friction depends upon the temperature and upon the value of the normal force, as well as upon the amount and kind of lubricant used.

It should be noted that the above law of static friction,  $F/N = \text{a constant}$ , is only approximately true in general, and that it is not even approximately true for very small unit normal forces or for very small area of contact of the surfaces.



## PROBLEMS

332. A plane on which a block of weight  $W$  rests is gradually tipped up until the block is on the point of sliding down the plane (Fig. 295). Show that

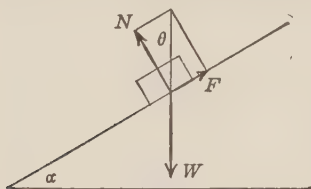


FIG. 295

when it is in this position the reaction of the plane upon the block is just equal and opposite to  $W$ , and that the angle of inclination of the plane is just equal to the angle of friction between the block and the plane.

333. A block of weight  $W$  is at rest on a plane inclined at an angle  $\alpha$  with the horizontal. It is acted upon by a force  $P$  inclined at an angle  $\phi$  with the plane, Fig. 296 (a). If  $P$  is just sufficient to start the block in motion up the plane, show that

$$P = \frac{W \sin (\alpha + \theta)}{\cos (\phi - \theta)},$$

where  $\theta$  is the angle of friction between the block and the plane.

(SUGGESTION. Note that the three forces that hold the block in equilibrium are  $W$ ,  $P$ , and the reaction,  $R$ , of the plane upon the block, and that these three forces must therefore form a closed triangle, Fig. 296 (b).)

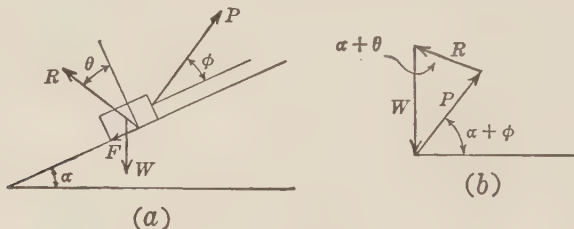


FIG. 296

334. Suppose the force  $P$  of Fig. 296 to be just sufficient to prevent the block from sliding down the plane. Show that in this case

$$P = \frac{W \sin (\alpha - \theta)}{\cos (\phi + \theta)}.$$

335. Prove that the least value of  $P$  that will just start the body in Fig. 296 up the plane is  $P = W \sin (\alpha + \theta)$ , and that this occurs when  $\phi = \theta$ .

[SUGGESTION. Notice that  $P$  in the formula of Problem 333 is least when the denominator of the fraction has its greatest value. Or, it is evident that  $P$  has its least value when it is perpendicular to  $R$ .]

336. Prove that the least value of  $P$  that will prevent the body of Fig. 296 from sliding down the plane is  $P = W \sin (\alpha - \theta)$ , and that this occurs when  $\phi = -\theta$ .

337. Find the value of a force  $P$  which is just necessary to start a weight of 100 lbs. up a plane inclined  $30^\circ$  to the horizontal, (a) when  $P$  is horizontal, (b) when  $P$  is parallel to the plane, (c) when  $P$  is inclined at an angle of  $15^\circ$  with



the plane, given that the coefficient of friction between the weight and the plane is 0.20. Find also the direction and magnitude of the least force that will just start the weight up the plane.

388. Solve the preceding problem if  $P$  is the force that will just prevent the body from sliding down the plane.

389. A weight  $W$  is raised by a horizontal force  $P$  which is applied to a wedge as shown in Fig. 297 (a). If the only friction is between the surfaces of the wedge and the weight, and the angle of friction there is  $\theta$ , prove that the value of  $P$  just necessary to start the wedge is

$$P = W \tan (\alpha + \theta).$$

(SUGGESTION. The triangles of forces that act on the wedge and on the weight are as indicated in Fig. 297 (b).)

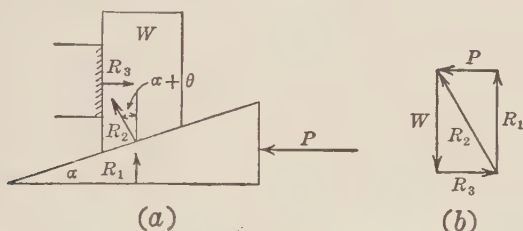


FIG. 297

390. Defining the *efficiency* of the wedge (Fig. 297) as the ratio of the work done against gravity in raising the weight, to the total work done by  $P$ , show that the efficiency,  $E$ , is given by the formula

$$E = \frac{\tan \alpha}{\tan (\alpha + \theta)},$$

and then show that for a given value of  $\theta$ ,  $E$  is a maximum when  $\alpha = 45^\circ - \theta/2$ . Show also that the maximum efficiency is equal to

$$\frac{1 - \sin \theta}{1 + \sin \theta}.$$

(Since a square-threaded screw used in raising a weight, the *jack-screw*, may be regarded as an inclined plane, this formula also holds for such a screw.)

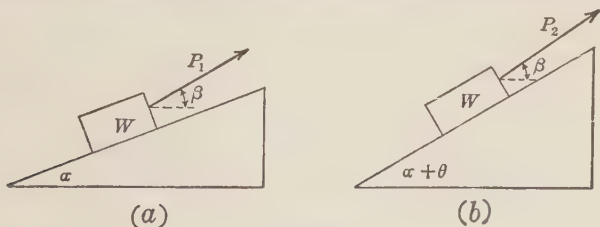


FIG. 298

391. In Fig. 298 are represented two inclined planes on each of which is a weight  $W$  acted upon by a force inclined at an angle  $\beta$  with the horizontal of

sufficient value to just start the weight up the plane. In Fig. 298 (a) the angle of inclination of the plane is  $\alpha$  and the angle of friction between the plane and the weight is  $\theta$ ; in Fig. 298 (b) the plane is smooth but the angle of inclination of the plane is greater than that of the plane in (a) by the angle  $\theta$ . Show that

$$P_1 = P_2 = \frac{W \sin (\alpha + \theta)}{\cos (\beta - \alpha - \theta)}.$$

392. In Fig. 299 the angles of friction at  $A$ ,  $B$ , and  $C$  are respectively  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ . Prove that the force  $P$  just sufficient to start the weight upward is

$$P = \frac{W \sin (\alpha + \theta_1 + \theta_2) \cos \theta_3}{\cos (\alpha + \theta_2 + \theta_3) \cos \theta_1},$$

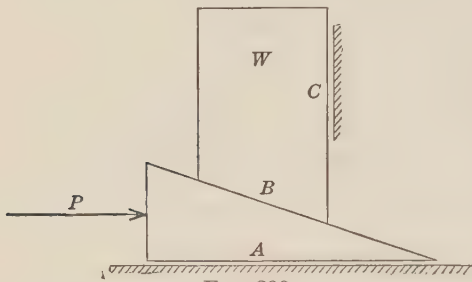


FIG. 299

assuming that the wedge is of negligible weight.

(SUGGESTION. Draw the triangles of forces for the weight and the wedge and apply the law of sines.) If  $W = 1000$  lbs.,  $\alpha = 20^\circ$ ,  $\theta_1 = 8^\circ$ ,  $\theta_2 = 10^\circ$ ,  $\theta_3 = 12^\circ$ , find the value of  $P$  to just raise the weight.

393. Show that in the preceding problem it would

be impossible to raise the weight if  $\alpha + \theta_2 + \theta_3 \geq 90^\circ$ .

394. Show that the value of  $P$  required to just prevent the weight of Problem 392 from descending is

$$P = \frac{W \sin (\alpha - \theta_1 - \theta_2) \cos \theta_3}{\cos (\alpha - \theta_2 - \theta_3) \cos \theta_1},$$

and then show that no force would be required to prevent the weight from descending if  $\alpha \leq \theta_1 + \theta_2$ .

**139. Kinetic Friction.** As has been stated in § 138, when two bodies in contact are in relative motion along the surfaces in contact, the friction is called *kinetic friction*. Kinetic friction has been found by experiment to depend upon several factors and does not follow any simple law as in the case of static friction. In most cases of kinetic friction in machines the rubbing surfaces are kept lubricated and it is found that the behavior of lubricated surfaces in regard to friction is very different from that of dry surfaces.

Defining the *coefficient of kinetic friction* as the ratio of the friction force on an element of area to the corresponding normal force, we may say that the coefficient of kinetic friction in the

case of lubricated surfaces depends upon the amount and kind of lubrication, the intensity of the normal force on the area in contact, the velocity of rubbing of one surface on the other, and the temperature. No general formula can adequately express its value.

In general, the coefficient of kinetic friction for two surfaces is less than the coefficient of static friction of the surfaces under the same conditions.

No attempt is made in this book to adequately discuss the subject of kinetic friction. The student is referred to textbooks on Machine Design, and to special articles in engineering magazines in which the results of experiments are published from time to time.<sup>1</sup>

**140. Pivot Friction.** For a given load on a pivot, a formula for the work done per revolution against friction can be derived only under certain assumptions. If we assume that the friction force  $dF$  on any element of area  $dA$  of a flat-end pivot (Fig. 300) to be proportional to the normal force  $dN$ , and let  $f = dF/dN$ , we have for the work done per revolution against friction,

$$U = \int 2\pi r' f dN,$$

where  $r'$  is the distance from the center to the element  $dA$  on which  $dN$  acts.

In the absence of definite knowledge as to how the normal pressure is distributed over the end of the pivot, it is customary to make either one of two hypotheses: (1) that the normal pressure is uniform over the end of the pivot, or (2) that the wear is uniform over the end of the pivot.

The first supposition would seem to be more nearly realized when the pivot is carried by a bath of oil under pressure, and the

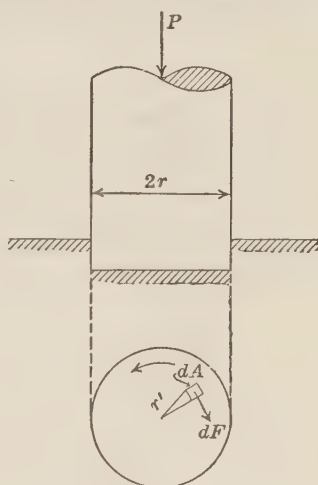


FIG. 300

<sup>1</sup> See also Goodman, *Mechanics Applied to Engineering*.

second supposition would seem to be more nearly correct when there is poor lubrication and a considerable wear occurs on the pivot end. In the first case, if  $P$  is the total load, and  $r$  the radius of the pivot, we have  $dN = (P/\pi r^2)dA$ . In the second case, we may assume that the wear on any element is proportional to the pressure and also to the distance rubbed through by the element. If then  $n$  is the normal pressure on a unit area at a distance  $r'$  from the center of the pivot, we may assume that  $nr' = C$ , or  $n = C/r'$  where  $C$  is a constant. The value of  $C$  can be determined by the equation

$$P = \int \frac{C}{r'} dA.$$

Under these suppositions the work done per revolution against the force of friction can be derived. The derivations of these formulas are asked for in the following problems.

### PROBLEMS

395. Prove that under the supposition of uniform pressure on the end of the flat-end pivot, the work done per revolution against friction is  $4\pi fP/3$ .

396. Prove that under the assumption of uniform wear, the work done per revolution against friction is  $\pi fP$ .

397. Show that if the pivot is hollow, or has a collar bearing as in Fig. 301, with inner and outer radii  $r_1$  and  $r_2$  respectively, the work done against friction per revolution is

$$\frac{4}{3} \pi f P \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$$

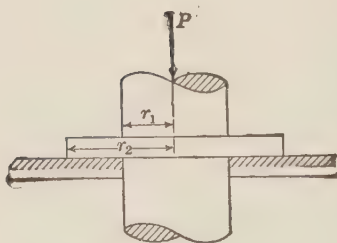


FIG. 301

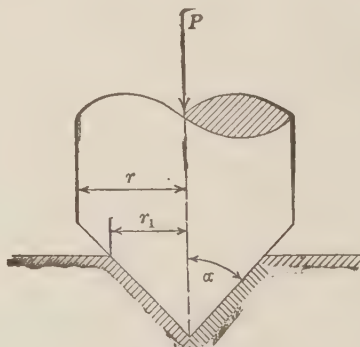


FIG. 302

under assumption (1) of § 140, and is

$$\pi f P (r_1 + r_2)$$

under assumption (2) of that article.

398. In the conical pivot of Fig. 302, assuming that the pressure is uniformly distributed over the conical rubbing surface, show that the work done per revolution against friction is

$$\frac{4}{3} \frac{\pi r_1 f P}{\sin \alpha}.$$

399. In the conical pivot of Fig. 302, assuming that the wear is constant over the surface, show that the work done per revolution against friction is

$$\frac{\pi f r_1 P}{\sin \alpha}.$$

400. A vertical shaft carrying a load of 25 tons is making 180 r.p.m. The shaft is 6 in. in diameter and has a flat-end bearing, which is well lubricated. If the coefficient of friction is 0.005, what hp. is lost due to friction, assuming the pressure to be uniform on the end of the pivot?

**141. The Friction Circle.** Consider a shaft that is being turned, or is on the point of being turned in a bearing, as in Fig. 303. Assume the bearing to be somewhat worn, so that as the shaft tends to turn in the bearing when carrying a vertical load, it will roll up on the bearing to some point, as  $A$ , and contact of shaft and bearing will be along a line through  $A$ . If  $F$  and  $N$  are respectively the friction and normal force exerted on the shaft by the bearing,  $f$  the coefficient of friction, and  $\theta$  the angle of friction, we have, by § 138,

$$\tan \theta = f = \frac{F}{N}.$$

Let  $r$  be the radius of the shaft and  $R$  the reaction of the bearing on the shaft.

A circle with center at the center of the shaft and tangent to the reaction of the bearing,  $R$ , is called the *friction circle*. Calling the radius of the friction circle  $r'$ , it is evident from the figure that

$$r' = r \sin \theta.$$

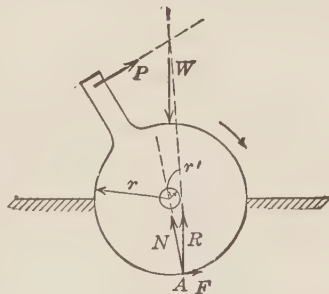


FIG. 303

**142. Rolling Friction.** Figure 304 represents a wheel of radius  $r$ , carrying a load  $W$ , being moved uniformly along a horizontal track by a horizontal force  $P$ . The wheel and track are supposed to be of hard material but they are, of course, slightly compressible.

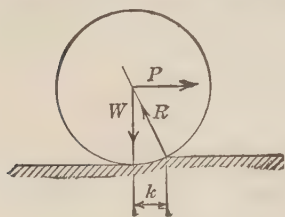


FIG. 304

Due to the compression of the wheel and track, the reaction of the track on the wheel will act at a point on the rim a little to the front of the center of the wheel, and a force  $P$  will be necessary to overcome the resistance offered

to the motion of the wheel by the track. Since the reaction,  $R$ , of the track, must balance the forces  $W$  and  $P$ , it must pass through the center of the wheel. Letting  $k$  be the distance that the point of application of  $R$  falls in front of the center of the wheel, we have approximately, since  $k$  is small compared to  $r$ , by taking moments about the point of application of  $R$ ,  $Pr = kW$ , or

$$P = \frac{kW}{r}.$$

The force  $P$  necessary to maintain the uniform motion of the wheel is called the *rolling friction*.

The quantity  $k$  is called the *coefficient of rolling friction*. Its value is expressed in inches. Hence, in computing the value of  $P$ , we must express  $r$  also in inches. The quantity  $k$  is not a coefficient in the usual sense of being the ratio of two quantities of the same kind.

**143. Belt Friction.** Let Fig. 305 (a) represent a belt in contact with a convex surface between the points  $A$  and  $B$ . At  $A$  and  $B$  the normals to the surface,  $AO$  and  $BO$ , are drawn and the angle  $AOB$  is called the *angle of contact* of the belt with the surface. The belt is supposed to be on the point of slipping, relative to the surface, from  $A$  toward  $B$ . Friction of the surface on the belt is then from  $B$  toward  $A$ , and the tension  $T_1$  at  $B$  is greater than the tension  $T_2$  at  $A$ .



Consider any extremely small portion  $CD$  of the belt of length  $\Delta s$ , and let the tensions at  $C$  and  $D$  be respectively  $T$  and  $T + \Delta T$ . Let  $\Delta F$  and  $\Delta N$  be the friction and normal forces exerted on this portion of the belt by the surface, Fig. 305 (b). Let the normals to the surface at  $C$  and  $D$  make angles  $\beta$  and  $\beta + \Delta\beta$  with the normal at  $A$ , and denote the radius of curvature of the surface at  $C$  by  $r$ . The angle which  $\Delta N$  makes with the normal,  $O'C$ , at  $C$  will be some fractional value of  $\Delta\beta$ , say  $k\Delta\beta$ , where  $k$  is between 0 and 1.

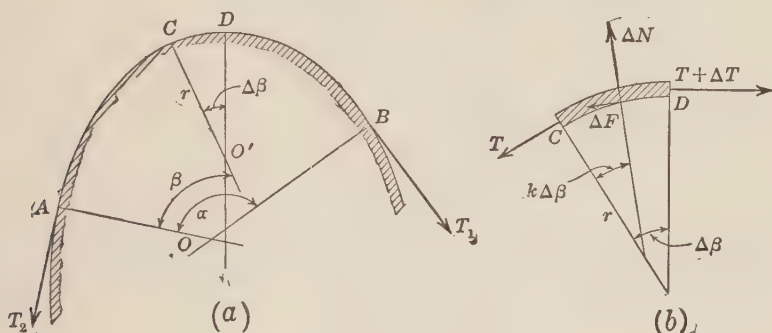


FIG. 305

Suppose now that the belt is in motion, as, for example, running on a pulley, and that its velocity is constant and equal to  $v$ . The tangential component of the acceleration of any point of the belt, as  $C$ , is then equal to 0 and the normal acceleration is equal to  $v^2/r$ . Let the mass per unit length of the belt be  $m$ . The mass of the portion  $CD$  is then  $m\Delta s$ , or  $mr\Delta\beta$ .

Taking summations of the forces acting on  $CD$  along the tangent and the normal at  $C$ , and regarding the mass of  $CD$  as that of a particle concentrated at  $C$ , we have, approximately,

$$(1) \quad (T + \Delta T) \cos \Delta\beta - T + \Delta N \sin (k\Delta\beta) - \Delta F \cos (k\Delta\beta) = 0,$$

$$(2) \quad (T + \Delta T) \sin \Delta\beta - \Delta N \cos (k\Delta\beta) - \Delta F \sin (k\Delta\beta) = m\Delta\beta v^2.$$

Dividing the first equation by  $\Delta F$  and passing to the limit as  $\Delta\beta$  approaches the limit zero, we obtain

$$(3) \quad dT = dF.$$

Dividing the second equation by  $\Delta\beta$  and passing to the limit, since  $\lim [(\sin \Delta\beta)/\Delta\beta] = 1$ , we find

$$(4) \quad T - \frac{dN}{d\beta} = mv^2.$$

If  $f$  is the coefficient of friction between the belt and the surface,

$$dN = \frac{dF}{f} = \frac{dT}{f},$$

and (4) may therefore be written in the form

$$T - mv^2 = \frac{dT}{fd\beta},$$

or, on separating the variables,

$$\frac{dT}{T - mv^2} = fd\beta.$$

Integrating this equation between the limits  $T_2$  and  $T_1$  for  $T$ , and 0 and  $\alpha$  for  $\beta$ , we have

$$\log \frac{T_1 - mv^2}{T_2 - mv^2} = f\alpha,$$

which may be written in the equivalent form

$$\frac{T_1 - mv^2}{T_2 - mv^2} = e^{f\alpha}.$$

If the belt is not in motion, but is on the point of slipping, the formula reduces to

$$\frac{T_1}{T_2} = e^{f\alpha}.$$

This latter equation may also be used when the belt velocity is small.

It should be noted that if the ft.-lb.-sec. system of units is used,  $m$  is the mass of a portion of the belt 1 ft. long, and  $v$  is the velocity of the belt in ft./sec.

**144. Power Transmitted by a Belt.** The work done per second against the friction  $dF$  acting on an element of the belt of length  $ds$ , by § 124, is  $vdF$ , and hence the total work per second done by

the belt against friction is  $v \Sigma dF$ . But in § 143 it was found that  $dF = dT$  and hence the work done per second by the belt against friction is

$$v \Sigma dT = (T_1 - T_2)v$$

and the horsepower transmitted by the belt is therefore

$$\text{hp.} = \frac{(T_1 - T_2)v}{550}.$$

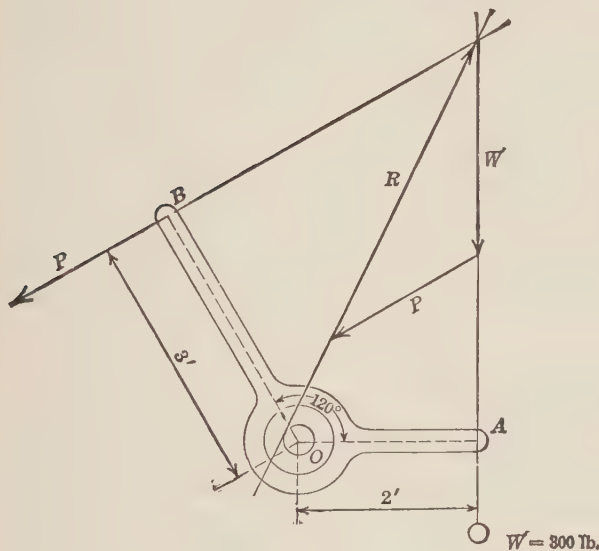


FIG. 306

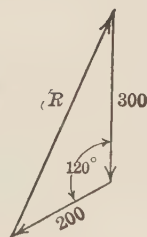


FIG. 307

**145. Illustrations.** EXAMPLE 1. In Fig. 306 the arm  $OA$  is horizontal and  $OB$  makes an angle of  $120^\circ$  with  $OA$ . The diameter of the fixed shaft on which the crank turns is 4 in. and the coefficient of friction between the shaft and the crank is 0.1. It is required to find the value of the force  $P$ , acting perpendicular to  $OB$  at a distance of 3 ft. from  $O$ , that will just raise the weight in the position shown.

The three forces acting on the crank are  $W$ ,  $P$ , and  $R$ , where  $R$  is the reaction of the axle. These three forces must be concurrent and must form a closed triangle. Since  $R$  must be tangent to the friction circle, the problem may be solved graphically by drawing a vector through the intersection of  $P$  and  $W$ , tangent to the friction circle, and to close the force triangle with the vectors of  $P$  and  $W$ . The value of  $P$  can then be scaled from the figure. The radius of the friction circle, by

§ 141, is

$$r' = r \sin \theta = \frac{1}{6} \sin (\tan^{-1} 0.1) = \frac{1}{6}(0.0995) = 0.0166 \text{ ft.} = 0.199 \text{ in.}$$

To determine on which side of the center  $R$  must be drawn, it may be noted that the reaction of the shaft on the crank must have its point of application on the upper part of the shaft, near the point where the line which joins the intersection of  $P$  and  $W$  to the center of the shaft cuts the circumference of the shaft, as in Fig. 306. With the direction of rotation given, the directions of  $N$  and  $F$  are determined and then  $R$  must be the resultant of  $F$  and  $N$ . This fixes approximately the line of  $R$  and determines on which side of the center of the shaft it passes. It is evident that in order to obtain an accurate solution, it is necessary to draw the figure to a large scale.

A solution may be obtained more easily by successive approximations as follows. If there were no friction, the value of  $P$ , obtained by moments, would be 200 lbs. The value of  $R$  could then be obtained from the triangle of forces, Fig. 307. Thus,

$$\begin{aligned} R^2 &= 200^2 + 300^2 + 2(200)(300)(0.5) = 190\,000, \\ R &= 435.9 \text{ lbs.} \end{aligned}$$

Then, since

$$F = R \sin \theta,$$

we have, approximately,

$$F = (435.9)(0.0995) = 43.37 \text{ lbs.}$$

Again taking moments, considering  $F$  to have the value just computed, we get a closer approximation to the value of  $P$ , namely,

$$P = \frac{1}{3}[(300)(2) + (43.37)\frac{1}{6}] = 202.41 \text{ lbs.}$$

Using this value instead of 200 in Fig. 307, the value of  $R$  is recomputed and found to be 437.8 lbs. With this value of  $R$ , the new trial value of  $F$  is

$$F = (437.8)(0.0995) = 43.56 \text{ lbs.}$$

Again taking moments to find  $P$ , we have

$$P = \frac{1}{3}[(300)(2) + (43.56)\frac{1}{6}] = 202.42 \text{ lbs.}$$

Since this is very nearly the same as the previous value found for  $P$ , it may be accepted as the correct value.

EXAMPLE 2. A car of total weight 25 tons is moving on a level track at 15 mi./hr. when the brakes are applied and the power turned off. Each of the eight wheels has a normal brake-shoe pressure of 3000 lbs. on each of two brake shoes, the centers of the shoes being on

opposite sides of the wheel and  $30^\circ$  below the horizontal center line, as indicated in Fig. 308. Each pair of wheels and their axle weighs 1450 lbs. and has a moment of inertia about the axis equal to 36. The radius of each wheel is 16 in. and the diameter of the axle is 5 in. The coefficient of brake friction is 0.15, the coefficient of axle friction is 0.004, and the coefficient of rolling friction is 0.02. How far will the car move before coming to rest?

**SOLUTION.** The total kinetic energy of the car at the instant the brakes are applied is equal to the work done against friction in bringing the car to rest. This work consists of that done against brake friction, that done against axle friction, and that done against rolling friction. Assume  $s$  to be distance in feet that the car will move. Each wheel will then make  $s/(2\pi r)$  revolutions, and if there is no slipping of the wheels on the track the surface of the wheel will move through  $[s/(2\pi r)](2\pi r)$  ft., or  $s$  ft., relatively to the brake shoe. The work done against the brake friction on all the wheels will then be

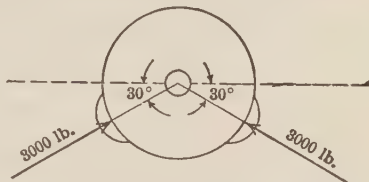


FIG. 308

$$(8)(2)(3000)(0.15)s = 7200s \text{ ft.-lbs.}$$

Subtracting the weight of the wheels and axles from the total weight of the car, we have  $50\,000 - 5\,800 = 44\,200$  lbs. for the weight resting on the axles, or an average<sup>1</sup> weight of 5 525 lbs. on each bearing. This is increased by the upward component of the brake-shoe pressure, which in this case amounts to  $(2)(3000) \sin 30^\circ$ , or 3000 lbs. on each bearing. The distance through which the axle friction acts is the distance that the circumference of the axle moves relatively to the bearing, or  $2.5s/16$  ft. The total work done against axle friction is therefore

$$(8)(8525)(0.004)(2.5s/16) = 42.625s \text{ ft.-lbs.}$$

The average force that each wheel exerts on the track is one eighth of the weight of the car, or 6250 lbs., and hence the total work done against rolling friction is

$$(8)(6250)(0.02)s/16 = 62.5s \text{ ft.-lbs.}$$

<sup>1</sup> We shall be able to infer from a later article that as the car is retarded the pressure on the front wheels is increased and that on the rear wheels is decreased. The total, however, is not changed, and hence the total friction is the same as if the force were equally divided on the wheels.

Hence the total work done against friction is

$$(7200 + 42.625 + 62.5)s = 7305s \text{ ft.-lbs. approximately.}$$

The kinetic energy of the car at the instant the brakes are applied is equal to  $Mv^2/2$  of the whole car plus  $I\omega^2/2$  of the wheels and axles. Since  $\omega = v/r = 3v/4 = 3(22)/4 = 16.5$  rad./sec., we have for the total kinetic energy of the car,

$$\frac{1}{2} \frac{50\,000}{32.2} (22)^2 + \frac{1}{2} (4)(36)(16.5)^2 = 395\,378 \text{ ft.-lbs.}$$

Equating the kinetic energy to the work done against friction, we have

$$\begin{aligned} 7305s &= 395\,378, \\ s &= 54.1 \text{ ft.} \end{aligned}$$

**EXAMPLE 3.** Figure 309 represents a friction brake for lowering a weight. Suppose the weight  $W$  to be 2000 lbs., the weight  $W'$  of the rotating drum and wheel to be 500 lbs., and to have a radius of gyration about its axis of rotation of 8 in., the radius  $r$  of the drum to be 5 in., and the radius  $R$  of the brake wheel to be 10 in. If the coefficient of brake friction is 0.25 and the angle of contact of the brake band with the wheel is  $180^\circ$ , what constant force  $P$  is required to cause the velocity of  $W$  to change from 30 ft./sec. to 5 ft./sec. in a descent of 10 ft., assuming that there is no axle friction?

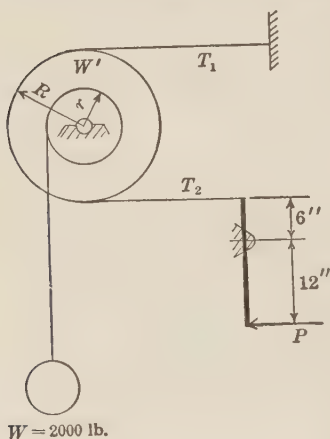


FIG. 309

**SOLUTION.** Here, as in § 144, the total friction between brake band and wheel is the difference between the tensions in the band at the tight and slack ends, namely,  $T_1 - T_2$ , and the work done against this friction while the weight descends 10 ft. is

$$(T_1 - T_2)10R/r \text{ ft.-lbs., or } 20(T_1 - T_2) \text{ ft.-lbs.}$$

The relation between  $T_1$  and  $T_2$  is (as shown in § 143)

$$T_1 = T_2 e^{0.25\pi} = 2.193T_2.$$

The relation between the velocity  $v$  of  $W$  and the angular velocity  $\omega$  of  $W'$  is

$$v = r\omega = 5\omega/12.$$



The work equation for the two bodies  $W$  and  $W'$  may then be written as follows,

$$(2000)(10) - (T_1 - T_2)(20) = \frac{1}{2} \frac{2000}{32.2} (5^2 - 30^2) + \frac{1}{2} \frac{500}{32.2} \left( \frac{8}{12} \right)^2 (5^2 - 30^2) \left( \frac{12}{5} \right)^2.$$

Replacing  $T_1$  by  $2.193T_2$  in this equation and solving it, we find

$$T_2 = 2706 \text{ lbs.}$$

Considering the equilibrium of the lever, we then find

$$P = T_2/2 = 1353 \text{ lbs.}$$

### PROBLEMS

401. In Fig. 306 find the value of  $P$  which will just prevent the 300-lb. weight from descending.

402. In Fig. 310,  $AB = BC = CD = 16$  in. The pins at  $A$  and  $C$  on which the member  $AC$  turns are each 4 in. in diameter. The pin at  $A$  is fixed in position. The pin at  $C$  is part of the member  $CD$ . The member  $CD$  is horizontal. The coefficient of friction at each of the pins is 0.25. By a graphical construction, making use of the friction circles at  $A$  and  $C$ , find the value of the force  $P$  applied at  $D$  which will just start the point  $C$  toward the right. Also make the construction assuming that there is no friction at the pins.

403. Find the value of  $P$  in Fig. 310 if  $C$  is on the point of moving toward the left.

404. Solve Problem 402 algebraically.

405. Assume the car of Example 2 of § 145 to have a velocity of 10 mi./hr., and the normal brake shoe pressure to be 2000 lbs. on each shoe. Find the distance the car will move before coming to rest after the power is shut off and the brakes are applied.

406. Figure 311 represents a transmission dynamometer, used to measure the power transmitted by a belt without absorbing an appreciable part of it.  $B$  and  $C$  are two idle pulleys, well lubricated, so that the change in belt tension in passing over these pulleys is negligible. Pulley  $D$  is the driver and  $E$  is the driven pulley. The diameter of pulley  $D$  is equal to the sum of the diameters of pulleys  $B$ ,  $C$ , and  $E$ , plus twice the thickness of the belt. The lever carrying pulleys  $B$  and  $C$  is pinned at  $A$  and the force  $P$  is applied to keep the lever  $BC$  perpendicular to the portions of the belt that are not in contact with the pulleys. Prove that

$$Pd = 2s(T_1 - T_2)$$

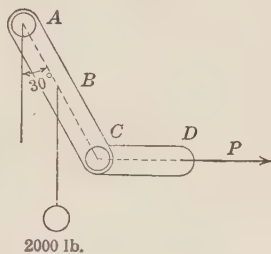


FIG. 310

and hence that if  $v$  is the velocity of the belt in feet per second, the power transmitted from pulley  $D$  to the pulley  $E$  is given by the formula

$$\text{hp.} = \frac{vPd}{1100s}.$$

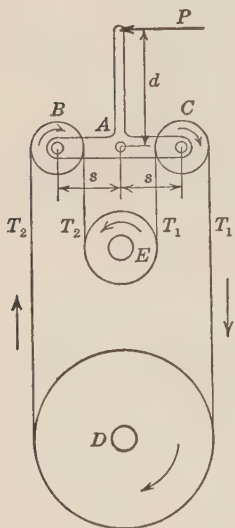


FIG. 311

407. Figure 312 represents a type of absorption dynamometer, or friction brake, used to absorb and measure the power developed by an engine or motor. A flywheel fastened to a shaft turned by the engine is surrounded by a flexible metal band to which are attached blocks of wood which rub against the rim of the wheel. The band is tightened until the wheel turns with constant number of revolutions per minute. The force  $P$  is measured by a platform scale. Show that if  $F$  is the total friction at the rim of the wheel,  $r$  the radius of the wheel in feet, and  $d$  the distance in feet from the center of the wheel to the line of action of  $P$ ,

$$Pd = Fr.$$

Show also that if  $n$  is the number of revolutions per minute made by the wheel, the power absorbed by the brake is given by the formula

$$\text{hp.} = \frac{2\pi nPd}{33\,000}.$$

408. A rope attached to a weight of 2000 lbs. is wound twice around a fixed horizontal shaft as represented in Fig. 313. If the coefficient of friction between rope and drum is 0.3, what tension  $T$  is necessary to prevent the weight from descending? What value of  $T$  would be required to raise the weight?

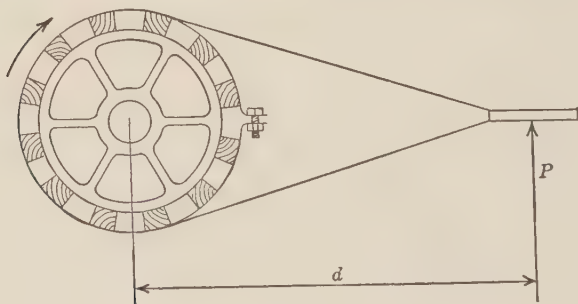


FIG. 312



FIG. 313

409. In Fig. 314 the radius  $r$  of the drum is 10 in., the radius  $R$  of the brake wheel is 20 in., the weight of the drum and brake wheel is 400 lbs., its radius of gyration with respect to the axis of rotation is 16 in., the weight  $W$  is 2000 lbs., the angle of contact of the brake band is  $270^\circ$ , and the coefficient of brake

friction is 0.25. What constant force  $P$  is required to change the velocity of  $W$  from 25 ft./sec. to 10 ft./sec. in a descent of 15 ft., assuming that there is no axle friction? Solve by using the work equation for the system.

410. Solve Problem 409 by using the equations

$$F = Ma, \quad \text{Torque} = I\alpha.$$

411. In Problem 409, assume that the diameter of the axle is 2 in. and that the coefficient of axle friction is 0.05, and solve for  $P$ .

(SUGGESTION. Use the values found in the solution of Problem 409 for  $T_1$  and  $T_2$ , the tensions in the belt, and the tension  $T$  in the cord attached to the weight, to find

an approximate value of the bearing on the axle. With this value compute the value of the friction on the axle. Then recompute  $T_1$  and  $T_2$  taking the axle friction into account. The computation may be repeated if necessary.)

412. The velocity of a belt which drives a pulley is 3600 ft./min., the tension in the tight side is 200 lbs. per in. of width, the coefficient of friction between the belt and pulley face is 0.25, and the weight of a portion of the belt 1 ft. long and 1 in. wide is 0.20 lbs. If the belt is on the point of slipping on the pulley and the angle of contact of the belt with the pulley is  $150^\circ$ , what is the tension in the slack side?

413. A pulley 4 ft. in diameter, making 240 r.p.m., drives a belt that transmits 25 hp. The belt is  $1/4$  in. thick and weighs 56 lbs. per cu./ft. The angle of contact is  $180^\circ$  and the coefficient of belt friction is 0.27. How wide must the belt be that the tension may not exceed 75 lbs. per inch of width?

414. Show that the formula for horsepower transmitted by a belt,

$$\text{hp.} = \frac{(T_1 - T_2)v}{550},$$

may be reduced to the form

$$\text{hp.} = \frac{(1 - e^{-f\alpha})(T_1 - mv^2)v}{550}.$$

415. Using the formula of Problem 414, show that with a maximum allowable tension  $T_1$ , the power transmitted is a maximum when the belt velocity is  $v = \sqrt{T_1/(3m)}$ .

416. What horsepower may be transmitted by a belt 6 in. wide,  $1/4$  in. thick, weighing 56 lbs. per cu./ft., when traveling 1500 ft./min., with angle of contact  $150^\circ$ , coefficient of belt friction 0.25, and having a maximum tension of 300 lbs. per sq. in. of cross-section of the belt?

417. Find the maximum horsepower that can be transmitted by the belt of Problem 416 with a maximum tension of 300 lbs. per sq. in. At what speed is this horsepower developed?

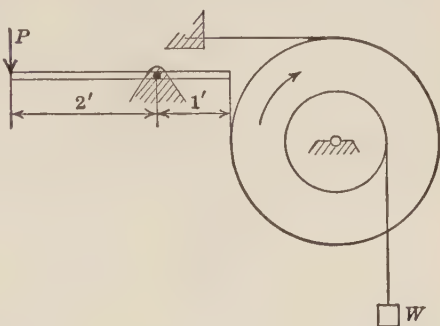


FIG. 314

**146. Conservative System of Forces.** Suppose that as a particle moves it is acted upon by a force whose value depends only upon the position of the particle, and further, that the components of this force parallel to the coordinate axes may be expressed as the corresponding partial derivatives of a single-valued function of the coordinates of the particle's position. Thus suppose  $U(x, y, z)$  to be a function of  $x, y$ , and  $z$  such that the force acting on a particle at  $(x, y, z)$  has components  $X, Y, Z$ , parallel respectively to the coordinate axes, with the values

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}.$$

Then, since

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz,$$

we have

$$dU = Xdx + Ydy + Zdz.$$

It was shown in § 127 that the work done on a particle in any displacement by a force with components  $X, Y, Z$  is

$$\int (Xdx + Ydy + Zdz),$$

the limits of integration being taken to correspond to the beginning and end of the displacement.

Let  $U_a$  and  $U_b$  be the values of  $U$  at the points  $A$  and  $B$  respectively. Then, since

$$Xdx + Ydy + Zdz = dU,$$

we have for the work done on the particle by the force when the position of the particle is changed from  $A$  to  $B$ ,

$$\text{Work} = \int_A^B dU = U_b - U_a.$$

This result depends only upon the initial and final positions of the particle.

Again, let there be a system of  $n$  particles which at any instant have coordinates  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$ ; and

further, let there be a function  $U$  of the coordinates of the points such that

$$\frac{\partial U}{\partial x_r} = X_r, \quad \frac{\partial U}{\partial y_r} = Y_r, \quad \frac{\partial U}{\partial z_r} = Z_r,$$

where  $X_r$ ,  $Y_r$ ,  $Z_r$  are the components of the force acting on the particle at  $(x_r, y_r, z_r)$ .

When the set of particles are changed from a position  $A$  to a position  $B$ , the work done on the particles by the set of forces will be

$$\int_A^B \sum_{r=1}^n (X_r dx_r + Y_r dy_r + Z_r dz_r).$$

But under the given conditions,

$$\begin{aligned} \Sigma (X_r dx_r + Y_r dy_r + Z_r dz_r) \\ = \Sigma \left( \frac{\partial U}{\partial x_r} dx_r + \frac{\partial U}{\partial y_r} dy_r + \frac{\partial U}{\partial z_r} dz_r \right) = dU. \end{aligned}$$

Hence, in the displacement of the set of particles from the position  $A$  to the position  $B$ , the work done on the particles by the set of forces is

$$\text{Work} = \int_A^B dU = U_b - U_a.$$

This, again, is a result which is independent of the paths taken by the particles in the displacement from the position  $A$  to the position  $B$ .

The function  $U$  is called the **force function**. A system of forces for which there exists a force function is called a **conservative system of forces**.

Since, by § 135, the work done on the system of particles is equal to their gain in kinetic energy, we have

$$U_b - U_a = E_b - E_a,$$

where  $E_a$  and  $E_b$  are the kinetic energy of the system of particles in the positions  $A$  and  $B$  respectively.

The negative of the force function is called the **potential energy** of the system. Then, since

$$E_a + (-U_a) = E_b + (-U_b),$$

we may say that in a system of particles under the action of a conservative system of forces, the sum of the kinetic energy and the potential energy of the particles remains constant in any displacement of the system.

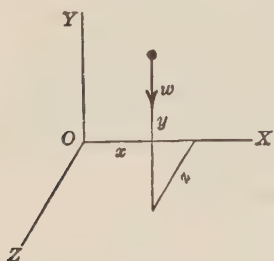


FIG. 315

**147. The Earth's Gravitational System Conservative.** That the earth's gravitational system of forces is conservative is proved as follows. Let  $w$  be the weight of any particle acted upon by gravity, and let coordinate axes be chosen so that the axis of  $y$  is vertical as in Fig. 315. The gravitational forces acting on the particle are then

$$X = 0, \quad Y = -w, \quad Z = 0.$$

A function which has these values as partial derivatives is seen by inspection to be

$$U = -wy + C.$$

If  $C$  is so chosen that  $U = 0$  when  $y = 0$ , the constant  $C = 0$ , and

$$U = -wy.$$

The potential energy of the particle is then  $wy$ .

If the particle is in motion subject to the gravitational force only, and if the velocity is  $v_0$  when  $y = 0$ , and is  $v$  when  $y = y$ , we may then write, since by § 146, the sum of the kinetic energy and the potential energy is constant,

$$wy + \frac{1}{2} \frac{w}{g} v^2 = 0 + \frac{1}{2} \frac{w}{g} v_0^2.$$

From this equation we obtain the familiar equation for the change in velocity of a particle under the action of gravity,

$$v_0^2 - v^2 = 2gy.$$

For a system of particles under the action of gravity, the force function is

$$U = -\sum wy = -W\bar{y},$$



where  $W$  is the weight of the system of particles and  $\bar{y}$  is the ordinate of the centroid of the particles measured from the horizontal plane from which the ordinates of the particles are measured.

**148. Spring Force Conservative.** Consider the force of compression in a spring acting on a body according to Hooke's law, Fig. 316. If we take the  $x$  axis along the axis of the spring, we have for the components of the force acting on the body,

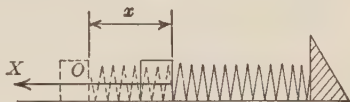


FIG. 316

$$X = -kx, \quad Y = 0, \quad Z = 0.$$

Therefore

$$U = -\frac{kx^2}{2} + C.$$

Since there is no work done by the force when  $x = 0$ , assuming the block to move from  $x = 0$  to  $x = x$ , the value of  $C$  is zero, and we have

$$U = -\frac{kx^2}{2}.$$

The potential energy of the block under the action of the spring force is then  $kx^2/2$ .

If  $v_1$  and  $v_2$  are the velocities of the block corresponding to the compressions in the spring  $x_1$  and  $x_2$ , and if  $M$  is the mass of the block, we have

$$\frac{kx_1^2}{2} + \frac{1}{2}Mv_1^2 = \frac{kx_2^2}{2} + \frac{1}{2}Mv_2^2.$$

### PROBLEMS

(NOTE.—The springs in the following problems are assumed to follow Hooke's law and to be without internal friction.)

418. Consider a spring with one end fixed, resting on a smooth horizontal plane, as in Fig. 316. Suppose it requires a force of  $k$  lbs. to compress the spring 1 ft. If the free end of the spring is attached to a weight  $W$  which can move along the plane, show by the conservation of energy, that if the weight is released from rest when the spring is compressed  $x_0$  ft., the velocity  $v$ , when the compression is  $x$  ft., is determined by the equation

$$\frac{kx_0^2}{2} = \frac{kx^2}{2} + \frac{1}{2} \frac{W}{g} v^2.$$

By differentiating this equation, show that the acceleration of the weight is proportional to  $-x$ , and that the motion of the weight is therefore simple harmonic motion (§ 100). Where is the center of oscillation of the weight? (This analysis takes no account of the kinetic energy of the spring, and is therefore not strictly correct.)

419. A spring of natural length  $l$  is extended a distance  $y_0$  when carrying a weight  $W$  hanging at rest, Fig. 317. If the weight is set in motion in the line

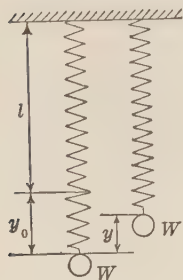


FIG. 317

of the spring and if at any instant  $v$  is the velocity and  $y$  is the distance of the weight above the position at which it hangs at rest, show by the theorem of the conservation of energy that

$$W y + \frac{1}{2} \frac{W}{g} v^2 + \frac{1}{2} \frac{W}{y_0} (y_0 - y)^2 = \frac{1}{2} \frac{W}{g} v_0^2,$$

where  $v_0$  is the velocity of the weight when  $y = 0$ .

By differentiating this equation, show that the acceleration of the weight is proportional to  $-y$ , and hence that the weight has simple harmonic motion.

420. If the coordinates of a moving particle of mass  $m_1$  are  $(x, y, z)$  referred to a fixed origin at which a particle of mass  $m_2$  is located, and if the force of attraction between the particles is

$$\frac{k m_1 m_2}{x^2 + y^2 + z^2},$$

show that the components of the attraction acting on  $m_1$  are

$$X = \frac{-k m_1 m_2 x}{(x^2 + y^2 + z^2)^{3/2}}, \text{ etc.}$$

and thence find a force function and show that the work done on  $m_1$  by the force of attraction when  $m_1$  moves from a distance  $r_1$  to a distance  $r_2$  from  $m_2$  is

$$k m_1 m_2 \left( \frac{1}{r_2} - \frac{1}{r_1} \right).$$

Hence find the work done on  $m_1$  when it is brought from an infinite distance to a distance  $r$  from  $m_2$ , under the force of attraction due to  $m_2$ .

**149. Non-Conservative Forces.** Forces for which a force function does not exist are not conservative. For example, the force of friction depends upon the direction of motion, and clearly the work done against friction acting upon a body when the body is moved from one position to another depends upon the path taken. Again, the resistance offered by a medium to a body moving through it usually depends upon the velocity of the body. The work done against this resistance when the body is moved from one position to another will then depend upon the velocity of the body as well as upon the path taken. Such forces are not conservative.

## CHAPTER VIII

### DYNAMICS OF RIGID BODIES

**150. D'Alembert's Principle.** Use has been made in § 129 of the principle due to D'Alembert. On account of its importance the argument is here repeated in somewhat different form.

A body in motion is acted upon by a set of external, or *impressed forces*. These forces are due to physical contact with other bodies, to gravitational forces, or to the attractions or repulsions of other bodies. The body is considered to be made up of elementary particles. On each of these particles certain forces act at each instant. These forces are called the *effective forces* acting on the particles. They are, of course, precisely those forces which if acting on the particles detached from the body would cause them to have just the motion that they have as parts of the body. These effective forces acting on any particle are made up of such of the body's impressed forces as act on that particle, and the internal reactions of the particles on each other. These latter forces are assumed to occur in pairs which are equal and opposite in the same line, so that when all of the particles of the body are considered, the system of internal reactions are in equilibrium among themselves. Then, since the whole set of effective forces acting on the particles is made up of all impressed forces and all internal reactions, it follows that the whole set of effective forces is equivalent to the set of impressed forces. If, then, on each particle of the body we should assume a force just equal and *opposite* to the effective force acting on that particle, we should have a set of *reversed effective forces* which would balance the set of impressed forces. Another form of the statement of D'Alembert's Principle may then be stated as follows. *The impressed forces acting on any body are in equilibrium with the set of reversed effective forces of the particles of the body.*

Application of this principle is made in the following paragraphs.

**151. Translation of a Rigid Body.** A body has pure translation if at each instant the velocities of all particles of the body are the same. It follows then that at any instant the accelerations

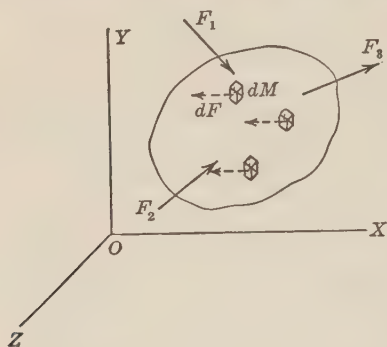


FIG. 318

of all particles of the body are the same. Suppose the body represented in Fig. 318 to have pure translation and that the acceleration of each particle of the body at a given instant is  $a$ , and let the body at the given instant be acted upon by the impressed forces  $F_1, F_2, F_3$ , etc. On each particle of mass  $dM$  assume a force  $dF = -dM \cdot a$ , the reversed effective force for that particle. Since these forces are

parallel and proportional to the masses of the particles at which they are applied, their resultant is a force,  $F$ , parallel and opposite to the acceleration and acting through the centroid of the body, such that

$$F = - \int a dM = - Ma,$$

where  $M$  is the mass of the body. Since, by D'Alembert's Principle, the resultant of the impressed forces must balance, the resultant of the reversed effective forces, it follows that the resultant of the impressed forces is a force acting through the centroid of the body, in the direction of the acceleration, and equal to the product of the mass of the body and the acceleration.

**152. Body Rotating about a Fixed Axis.** Assume a body under the action of a set of impressed forces,  $F_1, F_2, F_3$ , etc. to be rotating about a fixed axis (Fig. 319), here chosen as the axis of  $z$ . Assume that the angular velocity,  $\omega$ , and the angular acceleration,  $\alpha$ , are counted positive from  $OX$  toward  $OY$ . Let  $dM$ , with coordinates  $(x, y, z)$  be any element of mass of the body. This element will be moving in a circle in a plane parallel to the  $xy$  plane and with center,  $C$ , on the  $z$  axis. The effective force

required for the motion of this particle may be resolved into two components,  $r\omega^2 dM$  and  $r\alpha dM$ , the first directed toward  $C$  and the second along the tangent to the circle in the direction of  $\alpha$ . The reversed effective forces for the element are then  $dN = r\omega^2 dM$ , acting away from  $C$ , and  $dT = r\alpha dM$  along the tangent opposite to the direction of  $\alpha$ .

Applying D'Alembert's Principle, we may then write six equations which express the conditions of equilibrium of a set of forces, as follows:  $\Sigma X_i = 0$ ,

$\Sigma Y_i = 0$ ,  $\Sigma Z_i = 0$ ,  $\Sigma mom_x = 0$ ,  $\Sigma mom_y = 0$ ,  $\Sigma mom_z = 0$ . Using the subscripts  $i$  to indicate impressed forces and  $L_{ix}$ ,  $L_{iy}$ ,  $L_{iz}$  to represent the moments of the impressed forces about the  $x$ ,  $y$ , and  $z$  axes respectively, we have

$$\Sigma X_i + \int dN \cos \phi + \int dT \sin \phi = 0,$$

$$\Sigma Y_i + \int dN \sin \phi - \int dT \cos \phi = 0,$$

$$\Sigma Z_i = 0,$$

$$L_{ix} - \int zdN \sin \phi + \int zdT \cos \phi = 0,$$

$$L_{iy} + \int zdN \cos \phi + \int zdT \sin \phi = 0,$$

$$L_{iz} - \int rdT = 0.$$

Substituting  $dT = r\alpha dM$ ,  $dN = r\omega^2 dM$ ,  $\cos \phi = x/r$ ,  $\sin \phi = y/r$ , and noting that  $\int x dM = \bar{x}M$ ,  $\int y dM = \bar{y}M$ ,  $\int r^2 dM = I_z$ , we may write these equations in the form

$$(1) \quad \Sigma X_i + \omega^2 \bar{x}M + \alpha \bar{y}M = 0,$$

$$(2) \quad \Sigma Y_i + \omega^2 \bar{y}M - \alpha \bar{x}M = 0,$$

$$(3) \quad \Sigma Z_i = 0,$$

$$(4) \quad L_{ix} - \omega^2 \int yz dM + \alpha \int xz dM = 0,$$

$$(5) \quad L_{iy} + \omega^2 \int xz dM + \alpha \int yz dM = 0,$$

$$(6) \quad L_{iz} - \alpha I_z = 0.$$

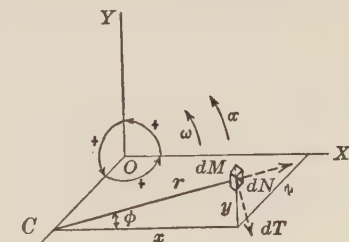


FIG. 319



These are the general equations which connect the impressed forces and the angular velocity and the angular acceleration of a body rotating about the  $z$  axis, at any instant of the motion. In taking moments about the coordinate axes, the positive directions have been chosen as indicated in Fig. 319; thus the moment about the  $x$  axis is counted from  $OY$  to  $OZ$ .

With certain choice of axes these formulas may be somewhat simplified. For example, if the  $xz$  plane is taken through the centroid of the body at the given instant, the terms in equations (1) and (2) that contain  $\bar{y}$  disappear from the equations.

**153. Resultant of the Reversed Effective Forces.** In equations (1) to (6) of the preceding article, the first term denotes a summation involving the impressed forces acting on the body, and the remaining terms the similar summation involving the reversed effective forces. Denoting the latter summations by the subscript  $e$ , choosing the  $xz$  plane to contain the centroid of the body, and letting the distance of the centroid of the body from  $OZ$  be  $\bar{r}$ , we may write

$$(1) \quad \Sigma X_e = \omega^2 \bar{r} M,$$

$$(2) \quad \Sigma Y_e = -\alpha \bar{r} M,$$

$$(3) \quad \Sigma Z_e = 0,$$

$$(4) \quad L_{ez} = -\omega^2 \int yz dM + \alpha \int xz dM,$$

$$(5) \quad L_{ey} = \omega^2 \int xz dM + \alpha \int yz dM,$$

$$(6) \quad L_{ez} = -\alpha I_z.$$

From the first three of these equations it follows that the set of reversed effective forces may be replaced by a force  $\omega^2 \bar{r} M$  parallel to the line which intersects the axis of rotation at right angles and passes through the centroid of the body, and a force  $-\alpha \bar{r} M$  perpendicular to the plane containing the axis of rotation and the centroid of the body. These two forces must be so placed that equations (4), (5), and (6) are also satisfied.



**154. Reversed Effective Forces for a Symmetric Rotating Body.**

Consider a body which is symmetric with respect to a plane perpendicular to the axis about which it is rotating. Let the axis of rotation be the  $z$  axis and the plane of symmetry the  $xy$  plane, with the  $x$  axis through the centroid of the body. The products of inertia,  $\int xz dM$  and  $\int yz dM$ , are then zero, and equations (4) and (5) of § 153 are satisfied if the resultant forces  $\omega^2 \bar{r} M$  and  $-\alpha \bar{r} M$  lie in the  $xy$  plane. Since a force acting in any line may be replaced by an equal force acting in a parallel line and a couple, we may replace the forces  $\omega^2 \bar{r} M$  and  $\alpha \bar{r} M$ , wherever they may be placed, by equal forces at the centroid and two couples in the plane of motion, and these two couples may then be combined into one couple in that plane. Let  $L$  be the moment of this resultant couple. Then in order that equation (6) be satisfied, we must have

$$L - \bar{r}(\alpha \bar{r} M) = -\alpha I_z,$$

or

$$L = -\alpha(I_z - \bar{r}^2 M).$$

But  $I_z = \bar{I} + \bar{r}^2 M$ , where  $\bar{I}$  is the moment of inertia of the body with respect to an axis through the centroid parallel to the axis of rotation. Hence

$$L = -\bar{I}\alpha.$$

The set of reversed effective forces are then equal to the two forces and the couple represented in Fig. 320.

If the impressed forces be combined with this set of forces the resulting system of forces will be in equilibrium.

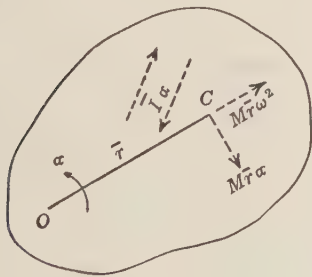


FIG. 320

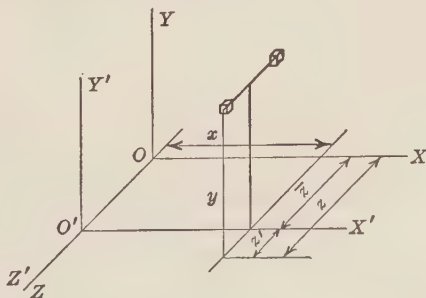


FIG. 321

**155. Product of Inertia of a Body Symmetric with Respect to a Plane Parallel to a Coordinate Plane.** Suppose a body to be symmetric with respect to a plane  $O'X'Y'$ , parallel to the  $xy$  plane, and let the centroid of the body be  $(\bar{x}, \bar{y}, \bar{z})$ , Fig. 321.

If an element of mass  $dM$  has coordinates  $x, y, z$ , referred to  $O - XYZ$  and coordinates  $x, y, z'$ , referred to  $O' - X'Y'Z'$ , then  $z = \bar{z} + z'$ .

The product of inertia  $\int yz dM$  then becomes

$$\int y(\bar{z} + z')dM = \int y\bar{z}dM + \int yz'dM.$$

Since the plane  $O'X'Y'$  is a plane of symmetry of the body, the term  $\int yz'dM$  is zero, and hence

$$\int yz dM = \int y\bar{z}dM = \bar{z} \int ydM$$

But  $\int ydM = \bar{y}M$ . Therefore  $\int yz dM = \bar{y}\bar{z}M$ .

In like manner it can be shown that

$$\int xz dM = \bar{x}\bar{z}M.$$

Of course it does not follow that  $\int xy dM$  is equal to  $\bar{x}\bar{y}M$  in this case.

**156. Illustrations. EXAMPLE 1.** A chest of weight  $W$  is dragged along a horizontal floor by a horizontal force  $F$  parallel to the floor at a distance  $h$  from the floor. The centroid is at a height  $c$  above the floor.

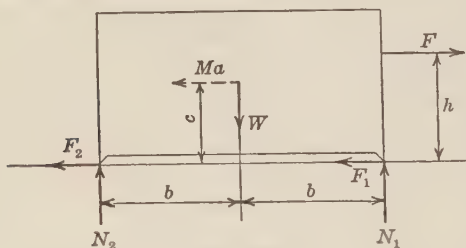


FIG. 322

The chest comes in contact with the floor only along the front and back edges and, on account of the symmetry, the reactions are regarded as acting in the vertical plane

containing the centroid, Fig. 322. The reactions are distant  $2b$  apart and at equal distances from the centroid. It is desired to find the normal and frictional forces at the points of contact with the floor, the coefficient of friction (the ratio of friction force to normal force) being  $f$ .

Since the body has pure translation, the resultant of all impressed forces must pass through the centroid, be parallel to the direction of

the acceleration, and be equal to the product of the mass and the acceleration (§ 151). Here the acceleration is clearly in the direction of the force  $F$ . Let the acceleration in this direction be  $a$ . Then a force  $Ma$ , opposite to the direction of  $a$ , and passing through the centroid of the body is the resultant of all the reversed effective forces, and hence forms with the impressed forces a set of forces in equilibrium. The set of forces is shown in Fig. 322, and we have for the conditions of equilibrium,

$$\Sigma X = 0; \quad F - Ma - F_1 - F_2 = 0,$$

$$\Sigma Y = 0; \quad W - N_1 - N_2 = 0,$$

$$\Sigma \text{Mom.} = 0; \quad 2bN_1 + cMa - Wb - hF = 0.$$

Also

$$F_1 = fN_1, \quad F_2 = fN_2, \quad M = W/32.2.$$

The solution of these equations will determine the values of  $a$ ,  $N_1$ ,  $N_2$ ,  $F_1$ , and  $F_2$ .

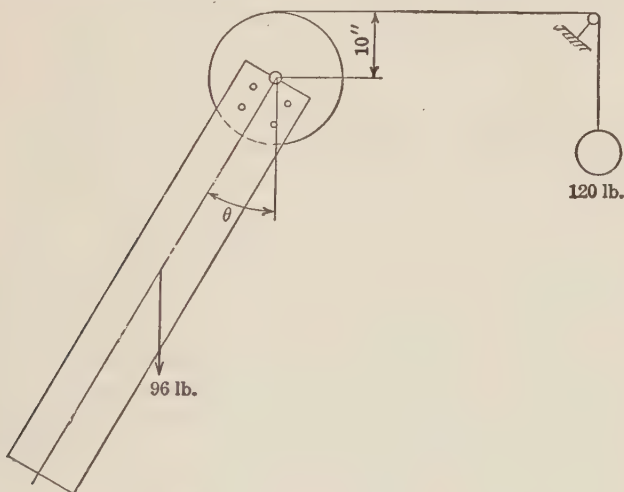


FIG. 323

EXAMPLE 2. A slab 6 ft. by 1 ft. weighing 96 lbs. is rigidly connected to a pulley of radius 10 in., weight 20 lbs., and having a radius of gyration of 8 in. with respect to its axis, as shown in Fig. 323. The pulley is mounted on a smooth pin. A cord attached to the rim of the pulley and wound round it runs horizontally from the pulley, passes over a smooth peg, and carries a weight of 120 lbs. at the other end. The weight is released from rest when the slab hangs vertically.

It is desired to find the reaction of the pin when the slab has turned through  $30^\circ$ .

The moment of inertia of the pulley and slab with respect to the axis of rotation is

$$\frac{96}{32.2} \left[ \frac{36 + 1}{12} + 9 \right] + \frac{20}{32.2} \left( \frac{4}{9} \right) = 36.30.$$

Then, letting  $T$  be the tension in the cord when the slab has turned through an angle  $\theta$ , we have for the motion of the rotating body and the 120-lb. weight the following equations respectively:

$$\frac{10}{12} T - (96)(3 \sin \theta) = 36.30\alpha,$$

$$120 - T = 120a/32.2,$$

where

$$a = 10\alpha/12, \quad \alpha \text{ clockwise.}$$

From these equations we find

$$(1) \quad 38.89\alpha = 100 - 288 \sin \theta,$$

or

$$38.89\omega d\omega = (100 - 288 \sin \theta) d\theta.$$

Integrating this equation between the limits 0 and  $\theta$  for  $\theta$ , and 0 and  $\omega$  for  $\omega$ , we get

$$(2) \quad 38.89\omega^2 = 200\theta - 576(1 - \cos \theta).$$

When  $\theta = 30^\circ$ , we find from (1) and (2),  $\alpha = -1.13$  rad./sec.<sup>2</sup>,  $\omega^2 = 0.708$  (rad./sec.)<sup>2</sup>,  $T = 123.5$  lbs.

To find the reaction of the pin on the pulley we now set out the body with the impressed forces that act upon it and also indicate the reversed effective forces, here represented by broken vectors. Instead of computing the reversed effective forces for the one rotating body, it is more convenient to compute them for the slab and pulley separately. Thus, for the pulley,

$$M\bar{r}\omega^2 = 0, \quad M\bar{r}\alpha = 0, \quad \bar{I}\alpha = (0.276)(-1.13) = -0.31 \text{ lbs.-ft.},$$

and for the slab

$$M\bar{r}\omega^2 = (2.98)(3)(0.708) = 6.33 \text{ lbs.},$$

$$M\bar{r}\alpha = (2.98)(3)(-1.13) = -10.10 \text{ lbs.},$$

$$\bar{I}\alpha = (9.19)(-1.13) = -10.38 \text{ lbs.-ft.}$$

The impressed forces acting on the body and the reversed effective

forces are then as indicated in Fig. 324, in which  $X$  and  $Y$  represent the components of the pin reaction on the pulley.

Applying the conditions for equilibrium, we have

$$-X + 123.5 - (10.10)(0.866) - (6.33)(0.5) = 0,$$

$$Y - 20 - 96 - (6.33)(0.866) + (10.10)(0.5) = 0.$$

Therefore

$$X = 111.6 \text{ lbs.}, \quad Y = 116.4 \text{ lbs.}$$

It was not necessary here to make any use of the reversed effective couples. However, it is useful as a check to see that the sum of the

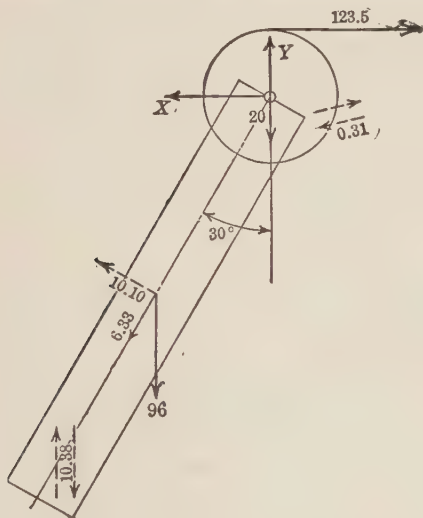


FIG. 324

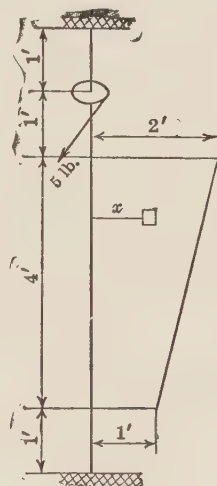


FIG. 325

moments of all the forces about some point is approximately zero. For example, taking moments about the center of the pulley, we have

$$(123.5) \left( \frac{10}{12} \right) + 0.31 + 10.38 - (96)(3)(0.5) + (10.10)(3) \\ = 102.92 + 0.31 + 10.38 - 144 + 30.30 = -0.09.$$

**EXAMPLE 3.** A thin uniform slab of trapezoidal form, of weight 60 lbs., is attached to a stiff vertical rod rotating in bearings under the action of a constant horizontal force of 5 lbs. applied to a cord wrapped around a pulley of radius 6 in. attached to the rod, as shown in Fig. 325. Suppose the slab to start from rest when the plane in which it lies is parallel to the line of the 5-lb. force and that it is desired to find the reactions of the two supports when the slab has made  $3/4$  of a revolution to the position shown in the figure. It is assumed that the lower

support takes all of the vertical force. (The angular velocity being small, the resistance of the air on the slab is neglected.)

In order to find the angular velocity and the angular acceleration, it is necessary to compute the moment of inertia of the body with respect to the axis of rotation. Letting  $m$  be the mass corresponding to a unit area of the face of the slab, the moment of inertia is

$$I = \int_0^4 \int_0^{(y/4)+1} mx^2 dy dx = \frac{m}{3} \int_0^4 \left( \frac{y}{4} + 1 \right)^3 dy$$

$$= \frac{m}{3} \left( \frac{y}{4} + 1 \right)^4 \Big|_0^4 = 5m.$$

But

$$m = \frac{60}{6(32.2)} = \frac{10}{32.2}; \quad \text{hence} \quad I = \frac{50}{32.2} = 1.553$$

Then

$$(5) \left( \frac{1}{2} \right) = \frac{50}{32.2} \alpha, \quad \text{or} \quad \alpha = 1.61 \text{ rad./sec.}^2$$

The work equation is

$$(5) \left( \frac{3\pi}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{2} \frac{50}{32.2} \omega^2, \quad \text{or} \quad \omega^2 = 15.17 \text{ (rad./sec.)}^2.$$

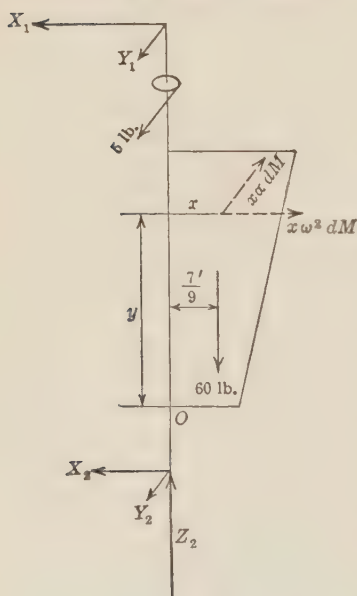


FIG. 326

Here the body is not symmetric with respect to a plane perpendicular to the axis of rotation, and the results of § 155 cannot be used. In this case we may consider the impressed forces acting on the body and the reversed effective forces of the separate particles as a system in equilibrium. The impressed forces are the weight of the slab acting at its centroid, which is easily found to be distant 7/9 ft. from the axis of rotation, the 5-lb. force at the rim of the pulley, and the components of the reactions of the bearings. These latter are  $X_1$  and  $X_2$  in the plane of the slab,  $Y_1$  and  $Y_2$  perpendicular to it, and  $Z_2$  vertical.

Considering now the impressed and reversed effective forces as a system in equilibrium, we may write



the following equations:

$$\begin{aligned}
 (1) \quad X_1 + X_2 &= \int x\omega^2 dM = \omega^2 \bar{x}M \\
 &= (15.17) \left( \frac{7}{9} \right) \left( \frac{60}{32.2} \right) = 21.99 \text{ lbs.}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad Y_1 + Y_2 + 5 &= \int x\alpha dM = \alpha \bar{x}M \\
 &= (1.61) \left( \frac{7}{9} \right) \left( \frac{60}{32.2} \right) = 2.33 \text{ lbs.}
 \end{aligned}$$

Taking moments about a line perpendicular to the slab, through the axis of rotation at the lower edge of the slab (through  $O$  in Fig. 326), we have

$$\begin{aligned}
 (3) \quad 6X_1 - X_2 - (60) \left( \frac{7}{9} \right) &= \int_0^4 \int_0^{(y/4)+1} m\omega^2 xy \, dy \, dx \\
 &= \frac{10}{32.2} (15.17) \left( \frac{34}{3} \right) = 53.39 \text{ lbs.-ft.}
 \end{aligned}$$

Taking moments about the lower edge of the slab, we have

$$\begin{aligned}
 (4) \quad 6Y_1 - Y_2 + (5)(5) &= \int_0^4 \int_0^{(y/4)+1} m\alpha xy \, dy \, dx \\
 &= \frac{10}{32.2} (1.61) \left( \frac{34}{3} \right) = 5.67 \text{ lbs.-ft.}
 \end{aligned}$$

Combining (1) and (3), we find

$$X_1 = 17.44 \text{ lbs.,}$$

$$X_2 = 4.55 \text{ lbs.}$$

Likewise, from (2) and (4), we get

$$Y_1 = -3.14 \text{ lbs.,}$$

$$Y_2 = 0.47 \text{ lbs.}$$

Summation of the vertical forces gives  $Z_2 = 60$  lbs.

The negative sign shows that the wrong direction has been assumed for  $Y_1$ .

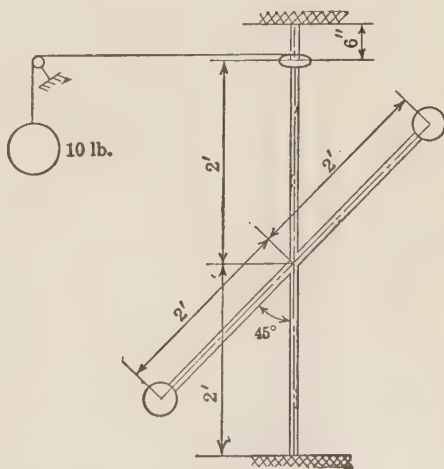


FIG. 327

EXAMPLE 4. Each of the two spheres of Fig. 327 weighs 75 lbs. and has a radius of 4 in. The rods connecting them with the axes are each approximately 1 ft. 7 in. long and each weighs 5 lbs. The pulley weighs 10 lbs., has a radius of 6 in. and a radius of gyration of 4 in. The vertical axis is solid, 2 in. in diameter and weighs 48 lbs. If the 10-lb. weight is released from rest when the centers of the spheres are in a vertical plane parallel to the cord, as shown in the figure, it is required to find the reactions of the supports when the system has made one complete revolution.

The moment of inertia of the rotating part is

$$\begin{aligned} & \left(\frac{1}{2}\right)\left(\frac{48}{32.2}\right)\left(\frac{1}{12}\right)^2 + \left(\frac{10}{32.2}\right)\left(\frac{1}{9}\right) + 2\left[\left(\frac{2}{5}\right)\left(\frac{75}{32.2}\right)\left(\frac{1}{9}\right)\right. \\ & \quad \left. + \left(\frac{75}{32.2}\right)(2)\right] + 2(0.707)^2\left(\frac{5}{32.2}\right)\left(\frac{19}{12}\right)^2\left(\frac{1}{3}\right) = 9.69 \end{aligned}$$

For the determination of  $\alpha$ , we then have the two equations

$$10 - T = \frac{10}{32.2} \frac{\alpha}{2}, \quad T\left(\frac{1}{2}\right) = 9.69\alpha,$$

from which we find  $\alpha = 0.512$  rad./sec.<sup>2</sup> and  $T = 9.92$  lbs.

The value of  $\omega^2$  is obtained from the work equation,

$$10\pi = \frac{1}{2} \frac{10}{32.2} \frac{\omega^2}{2} + \frac{1}{2} (9.69)\omega^2,$$

from which we find

$$\omega^2 = 6.432 \text{ (rad./sec.)}^2.$$

The values of the reactions will now be found by use of the equations of § 152. In using these equations, the axis of rotation must be taken as the  $z$  axis. In order to have the same relative position of the axes as was used in deriving the equations, the  $x$  axis is chosen perpendicular to the plane containing the centers of the spheres and the  $y$  axis is chosen in that

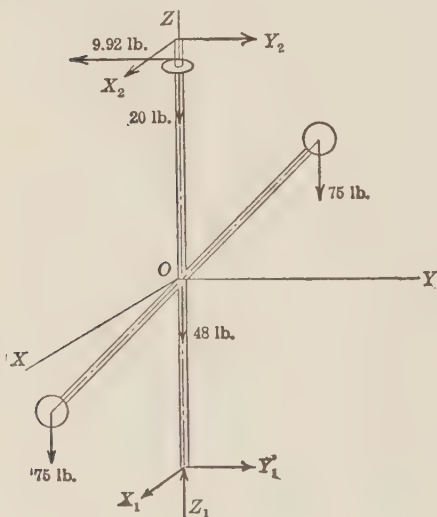


FIG. 328

plane. The origin is conveniently chosen at the intersection of the rod and the axle. The impressed forces are then as shown in Fig. 328. Before substituting in equations (4) and (5), it is necessary to compute the values of  $\int yz dM$  and  $\int xz dM$  for the rods which connect the spheres with the shaft. Since the rods are symmetric with respect to the  $yz$  plane, the value of  $\int xz dM$  is zero. Also for the rod on the right of the shaft,

$$\int yz dM = \frac{1}{32.2} \frac{5}{l} \int_0^l (0.707)^2 l^2 dl = \frac{5}{32.2} \frac{1}{2} \frac{l^2}{3} = \frac{5}{6} \frac{1}{32.2} \left( \frac{19}{12} \right)^2 = 0.065$$

The same result is obtained for the other rod, since in evaluating the integral both  $y$  and  $z$  of the element are changed in sign only. The value of  $\int xz dM$  for each sphere is zero and of  $\int yz dM$  is

$$\frac{75}{32.2} (1.414)(1.414) = 4.66 \quad (\S 155).$$

In substituting for  $\bar{x}M$ , the sum of the  $\bar{x}M$ 's of the various parts may be used instead of  $\bar{x}M$  of the whole body. In this problem both  $\bar{x}M$  and  $\bar{y}M$  are zero.

Equations (1) to (5) are then as follows:

- $$\begin{aligned} (1) \quad & X_1 + X_2 = 0, \\ (2) \quad & Y_1 + Y_2 - 9.92 = 0, \\ (3) \quad & Z_1 - 218 = 0, \\ (4) \quad & 2Y_1 - 2.5Y_2 + (9.92)(2) - 6.432[(2)(0.065) + (2)(4.66)] = 0, \\ (5) \quad & -2X_1 + 2.5X_2 - (0.512)[(2)(0.065) + (2)(4.66)] = 0. \end{aligned}$$

From these equations we find  $X_1 = 1.08$  lbs.,  $X_2 = -1.08$  lbs.,  $Y_1 = 14.61$  lbs.,  $Y_2 = -4.69$  lbs.,  $Z_1 = 218$  lbs.

### PROBLEMS

NOTE. In the following problems no account is to be taken of air resistance.

421. In Example 1 of § 156, suppose the chest to weigh 500 lbs., the distance between the points of contact with the floor to be 4 ft., the height of the centroid above the floor to be 1.5 ft., the coefficient of friction to be  $1/4$ , and the value of  $F$  to be 150 lbs.; find the values of  $N_1$  and  $N_2$  when (a)  $h = 1$  ft., (b)  $h = 2$  ft. For what value of  $h$  would  $N_1 = N_2$ ?

422. A door of weight 100 lbs. is suspended as shown in Fig. 329 and is moved by the horizontal force of 20 lbs., applied as shown.

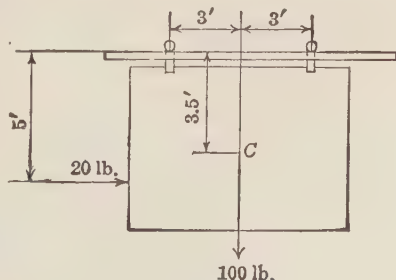


FIG. 329

If at each roller the horizontal force is  $1/8$  of the vertical force at that point, find the values of the reactions at the rollers and the acceleration of the centroid of the door.

423. A rectangular block of weight  $W$ , height  $h$ , and square base of side  $b$ , rests upon a moving platform which has a horizontal acceleration  $a$ , Fig. 330. If the block touches the platform only along the front and rear edges and is

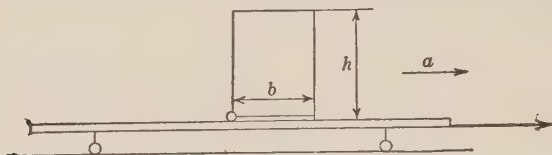


FIG. 330

hinged to the platform along the rear edge, show that the reaction,  $N_1$ , at the front edge, and the vertical component,  $N_2$ , of the reaction at the hinge, are respectively

$$N_1 = \frac{W}{2} - \frac{Wha}{2bg}, \quad N_2 = \frac{W}{2} + \frac{Wha}{2bg}.$$

For what value of  $a$  will  $N_1$  reduce to zero?

424. A cylindrical block of uniform material, of height  $h$  and radius of base  $r$ , stands upon a horizontal platform which moves along a straight track. If the surface is rough enough to prevent the block from slipping on the platform, what acceleration may be given to the platform before the block overturns?



FIG. 331

425. A uniform rod of length  $l$ , small cross-sectional area  $A$ , and mass  $M$  is attached to a shaft of radius  $r$  at right angles to the shaft. If the shaft is rotating with constant angular velocity  $\omega$ , show that (neglecting the force of gravity on

the rod) the rod is in pure tension and that the maximum tension per unit area of the cross-section is  $M(l + 2r)\omega^2/2A$ .

426. Find the force with which one half of a solid uniform disk of mass  $M$  and radius  $r$  pulls on the other half when rotating about an axis through its center, perpendicular to the face of the disk, with angular velocity  $\omega$ .

427. A slender uniform rod of length  $l$  and weight  $W$  is supported in a horizontal position by a vertical string at one end and a smooth pivot at the other, Fig. 331. If the string is cut, show that the pivot reaction changes suddenly from  $W/2$  to  $W/4$ . Find

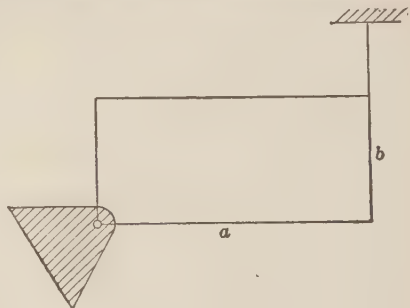


FIG. 332

the horizontal and vertical components of the pivot reaction when the rod has swung through  $60^\circ$ , through  $90^\circ$ .

428. A uniform rectangular slab of weight  $W$  and dimensions  $a$ ,  $b$ , and  $c$  is supported by a smooth pin along an edge  $c$  and by a vertical thread at the opposite edge so that the face containing the edges  $a$  and  $c$  is horizontal, Fig. 332. What is the reaction at the pin? If the thread is cut, prove that the horizontal and vertical components of the pin reaction suddenly become

$$H = \frac{3abW}{4(a^2 + b^2)}, \quad V = \frac{(a^2 + 4b^2)W}{4(a^2 + b^2)}.$$

Find also the  $H$  and  $V$  components of the pin reaction when the slab has turned through  $90^\circ$ .

429. If the slab of the preceding problem is pivoted at the center of the side  $b$  and released from rest when the slab is above the pivot with the edge vertical, find the  $H$  and  $V$  components of the pin reaction when the slab has turned through an angle  $\theta$ .

430. A slab of weight  $W$  has one face 4 ft. by 1 ft. and is pivoted on a smooth horizontal pin perpendicular to this face and passing through the center of a 1 ft. edge. If the slab starts from rest when its centroid is vertically above the pin, show that when it has turned through an angle  $\theta$  the values of the angular acceleration and angular velocity are given by  $\alpha = 24g \sin \theta / 65$ ,

$$\omega^2 = 48g(1 - \cos \theta) / 65.$$

Suppose now that the upper half of the slab is connected with the lower half by a pin at one edge and a very short link at the other and that the two halves have no other contact. The impressed forces acting on the upper half could then be represented by  $F_1$ ,  $F_2$ ,  $F_3$ , and  $W/2$ , as in Fig. 333. The reversed effective forces are also sketched in as broken vectors.

Show that for the upper half of the slab,

$$\bar{I}\alpha = \frac{W \sin \theta}{13},$$

$$M\bar{r}\alpha = \frac{36W \sin \theta}{65},$$

$$M\bar{r}\omega^2 = \frac{72W(1 - \cos \theta)}{65}$$

and then prove that

$$F_1 = W \left( \frac{209 \cos \theta}{260} - \frac{17 \sin \theta}{130} - \frac{36}{65} \right),$$

$$F_2 = W \left( \frac{209 \cos \theta}{260} + \frac{17 \sin \theta}{130} - \frac{36}{65} \right).$$

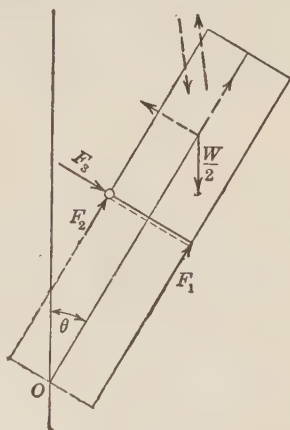


FIG. 333

Show that for the values of  $\theta$  written below, the corresponding values of  $F_1$  and  $F_2$  are as indicated:

| $\theta$ | 0     | 30°    | 45°      | 60°      | 90°      |
|----------|-------|--------|----------|----------|----------|
| $F_1$    | 0.25W | 0.077W | - 0.078W | - 0.265W | - 0.684W |
| $F_2$    | 0.25W | 0.208W | 0.108W   | - 0.039W | - 0.423W |

Notice that  $F_1$  changes from compression to tension before  $F_2$  does.

Consider these results in connection with the way in which a tall chimney breaks in falling.

431. A slender rod  $CD$  of weight  $W$  is attached rigidly to a shaft  $AB$  at an angle  $\theta$  with the shaft and is rotating about  $AB$  with angular velocity  $\omega$  and angular acceleration  $\alpha$ , Fig. 334. Show that the resultant of the reversed effective forces on the elements of mass of  $CD$  has one component  $Wl \sin \theta \omega^2/2g$  in the plane of  $AB$  and  $CD$ , perpendicular to  $AB$  and at a distance  $2l \cos \theta/3$  from the point of attachment,  $C$ , and a second component  $Wl \sin \theta \alpha/2g$  perpendicular to the plane of  $AB$  and  $CD$  in a line which cuts  $CD$  at a point two-thirds of the distance from  $C$  to  $D$ .

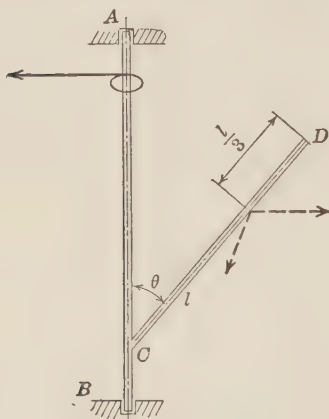


FIG. 334

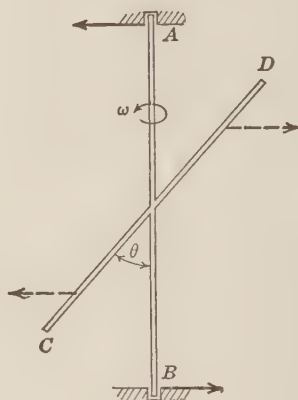


FIG. 335

432. A slender rod  $CD$  of length  $l$  and weight  $W$  is attached at its middle point to a small shaft  $AB$  at an angle  $\theta$  with the shaft, Fig. 335. When the shaft is rotating with constant angular velocity  $\omega$ , show that there is introduced a couple of moment  $Wl^2 \sin \theta \cos \theta \omega^2/12g$  tending to make the shaft wobble.

433. A thin, uniform, triangular slab with dimensions as shown in Fig. 336 is attached to a small shaft and is rotating with the shaft with angular velocity  $\omega$  and angular acceleration  $\alpha$ . Show that the resultant of the reversed effective forces has two components: one,  $Wb\omega^2/3g$ , in the plane of the slab, at right angles to  $AB$  and in a line distant  $3h/4$  from  $C$ ; the other,  $Wb\alpha/3g$ , perpendicular to the plane of the slab, in a line distant  $b/2$  from  $AB$  and at a height  $3h/4$  above  $C$ .



434. The system of Fig. 337 starts from rest when in the position shown, with the horizontal string parallel to the plane of the slab. When the slab has made two revolutions, what are the reactions of the supports? (Use reversed effective forces as found in the preceding problem).

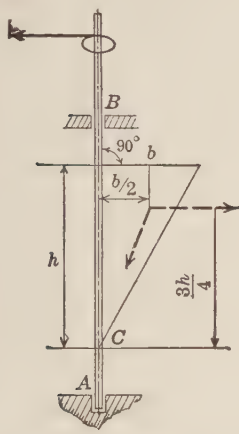


FIG. 336

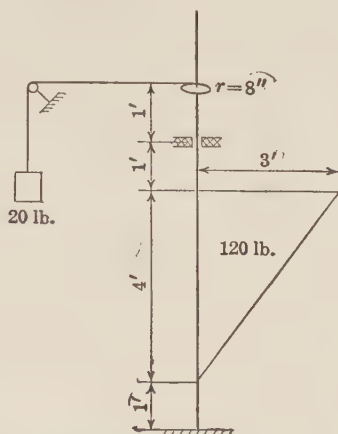


FIG. 337

435. The spheres *C* and *D* of Fig. 338 weigh respectively 120 lbs. and 50 lbs., and their radii are respectively 4 in. and 3 in. The planes containing

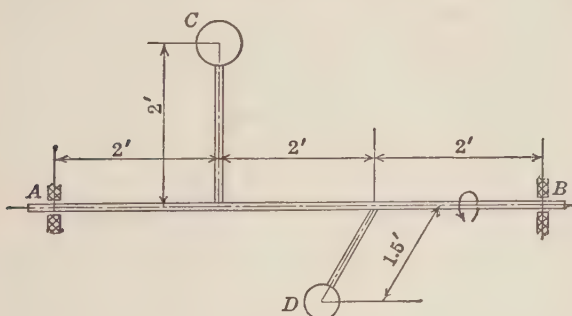


FIG. 338

their centers and the axis of rotation are  $90^\circ$  apart. At a given instant, when *C* is directly above the axis the angular velocity is  $4\pi$  rad./sec. in the direction indicated in the figure. If no forces are acting except gravity and the reactions of the smooth bearings at

*A* and *B*, find the horizontal and the vertical components of the reactions of the bearings after the shaft has turned through  $90^\circ$  from the given position; also when it has turned through  $150^\circ$ . Consider the weight of the arms, connecting the spheres to the shaft, to be negligible.

436. A thin rigid circular disk of mass  $M$  and radius  $r$  ft. is mounted on a small shaft, with its plane making an angle  $\theta$  with a plane perpendicular to the shaft. Show that when the shaft is rotating with constant angular velocity  $\omega$  rad./sec. there is introduced a couple, tending to make the shaft wobble, of moment  $M r^2 \sin \theta \cos \theta \omega^2 / 4$  ft.-lbs., or, if  $\theta$  is small, of  $M r^2 \theta \omega^2 / 4$  ft.-lbs., nearly.

**157. Resultant of the Reversed Effective Forces in the Plane Motion of a Body which is Symmetric with respect to the Plane of Motion.** If any filament of the body perpendicular to the plane of motion is considered, it is evident that all points in this filament at any instant have the same acceleration, parallel to the plane of motion, and hence that the effective forces on the elements of this filament are parallel and proportional to the masses of these

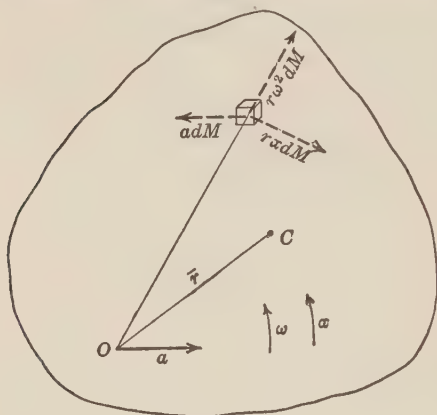


FIG. 339

elements. The resultant of these effective forces on the elements of the filament acts in the plane of motion through the centroid of the filament.

In Fig. 339, let  $dM$  represent the mass of any filament of infinitesimal cross-section perpendicular to the plane of motion. Suppose a point  $O$  in the plane of motion to have an acceleration  $a$ ,

and let the angular velocity and angular acceleration be  $\omega$  and  $\alpha$  respectively. The acceleration of  $dM$ , at distance  $r$  from  $O$ , is then made up of three components,  $a$ ,  $r\omega^2$ , and  $r\alpha$ , as in § 117. The reversed effective forces of the filament are then  $adM$ ,  $r\omega^2 dM$ , and  $r\alpha dM$ , opposite in directions to the components of acceleration of  $dM$ . These are represented as broken vectors in the figure.

The set of forces  $adM$  for all the elements were shown in § 151 to have a resultant  $Ma$ , acting through the centroid of the body, and the other sets of forces,  $r\omega^2 dM$  and  $r\alpha dM$ , were shown in § 154 to have resultants  $M\bar{r}\omega^2$  and  $M\bar{r}\alpha$ , acting through the centroid of the body, and a couple of moment  $\bar{I}\alpha$  in the plane of motion. If then  $C$  is the centroid of the body,  $\bar{r}$  its distance from  $O$ , a point in the plane of motion with acceleration  $a$ , the system of reversed effective forces is equivalent to those indicated by broken vectors in Fig. 340.

We see from this figure that the sum of the moments of the reversed effective force about a centroidal axis perpendicular to the plane of motion is equal to  $-\bar{I}\alpha$ . Since the impressed forces and the reversed effective forces are in equilibrium, it follows that in the case of a body having plane motion, the sum of the moments of the impressed forces about a centroidal axis perpendicular to the plane of motion is equal to the product of the moment of inertia of the body about that axis and the angular acceleration of the body.

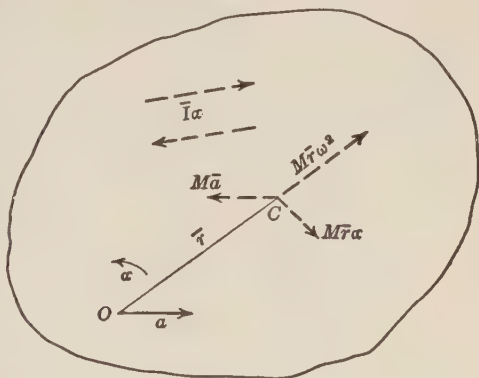


FIG. 340

**158. Applications.** **EXAMPLE 1.** A wheel of radius 1.5 ft., weight 200 lbs., and radius of gyration 1.25 ft. with respect to an axis through the center perpendicular to the plane of the wheel, is pulled up a plane inclined  $20^\circ$  to the horizontal by a force of 100 lbs., inclined  $15^\circ$  to the plane and applied through the center of the wheel. The plane is sufficiently rough to prevent slipping of the wheel on the surface. It

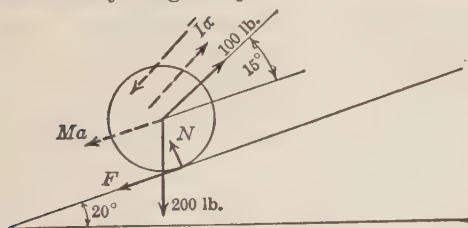


FIG. 341

is required to find the acceleration of the center of the wheel and the normal and frictional forces exerted by the plane on the wheel.

We may here conveniently take the center of the wheel as the point

$O$  of § 157. Then  $\bar{r}$  becomes zero, and the reversed effective forces are  $Ma$ , at the center, opposite to the acceleration,  $a$ , of the center, and a couple  $\bar{I}\alpha$  in the plane of motion, opposite in direction to  $\alpha$ . These and the impressed forces are shown in Fig. 341. By applying the conditions of equilibrium, and noting that

$$1.5\alpha = a,$$

$$M = 200/32.2 = 6.21,$$

$$I = (6.21)(1.25)^2 = 9.71,$$

we have

$$(1) \quad 100 \cos 15^\circ - 6.21a - F - 200 \sin 20^\circ = 0,$$

$$(2) \quad 100 \sin 15^\circ + N - 200 \cos 20^\circ = 0,$$

$$(3) \quad 9.71a/1.5 - 1.5F = 0.$$

From these equations we find  $a = 2.68$  ft./sec.<sup>2</sup>,  $F = 11.57$  lbs.,  $N = 162.1$  lbs.

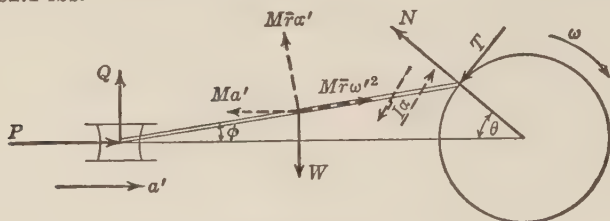


FIG. 342

EXAMPLE 2. In Fig. 342 are represented the impressed forces acting on a connecting rod, together with the reversed effective forces.

Using the notation of Problem 344, having given the angular velocity,  $\omega$ , of the crank-pin, here assumed constant, the values of  $\omega'$ ,  $\alpha'$ , and  $a'$  are found, either by graphical construction, as in Problem 344, or by computation, as in Problem 345. Taking the cross-head end of the connecting rod as the point  $O$  of § 157, the vectors representing the reversed effective forces are placed on the figure as in Fig. 342. These together with the impressed forces form a system in equilibrium. By writing and solving the three equations of equilibrium, the three reactions at the ends of the connecting rod can be determined. It is assumed that the cross-head guide is smooth, and hence that the reaction,  $Q$ , is normal to the path of the cross-head. At the crank end of the connecting rod, the reaction is represented as two components,  $N$  and  $T$ , normal and tangent to the path, respectively.

### PROBLEMS

437. After the wheel of Fig. 341 moves up the plane for 10 seconds under the action of the forces shown, the 100-lb. force is suddenly replaced by one of 20 lbs. in the same line. Show that the direction of action of the frictional force,  $F$ , is immediately reversed. Find then the new values of  $F$ ,  $N$ , and  $a$ . How much farther up the plane will the wheel move?

438. A uniform connecting rod is 6 ft. long between bearing centers, its weight is 300 lbs., and its moment of inertia with respect to a line through its centroid perpendicular to its plane of motion is 27. The radius of the crank-pin circle is 15 in. If the crank is making 120 r.p.m. (constant) and the steam pressure ( $P$  of Fig. 342) is 6000 lbs. when the angle  $\theta$  is  $30^\circ$ , find the values of  $Q$ ,  $N$ , and  $T$  in that position of the connecting rod.

439. A disk has one end of a fine thread fastened to a point in its rim and the thread is wound round the disk. The other end of the thread is attached to a point in the surface of a smooth plane inclined  $\theta$  to the horizontal and the disk is released and allowed to roll down the plane subject to the tension in the thread, Fig. 343. Show that the values of  $T$ , the tension in the thread, and  $N$ , the normal force against the disk, are  $T = W \sin \theta/3$ ,  $N = W \cos \theta$ . Show then that if  $\theta$  becomes  $90^\circ$ , the value of  $T$  becomes  $W/3$ , the value of  $N$  becomes 0, and the acceleration of the center becomes  $2g/3$ . Compare the motion with that of a disk falling under the action of gravity and the tension in a light string wound round the circumference and fastened as in Fig. 344.

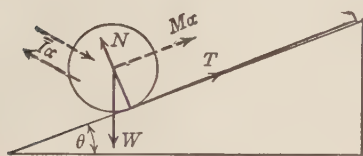


FIG. 343



FIG. 344

159. **Balancing a Shaft.** When a shaft runs at high speed it is necessary to have it accurately balanced so that large reactions at the bearings will not be introduced. The case will be discussed here of a shaft rotating with uniform angular velocity about a horizontal axis supported in two bearings.

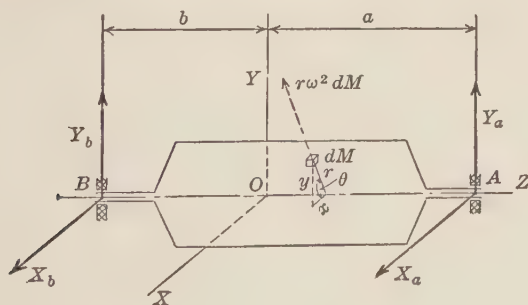


FIG. 345

Let Fig. 345 represent any rotor, assumed to be rigid, supported in bearings A and B, distant  $l$  apart, rotating with constant angular velocity  $\omega$ . Assume a set of rectangular axes with origin O at distances  $a$  and  $b$  from A and B respectively.



Let the axis of  $z$  be the axis of rotation and let the  $x$  and  $y$  axes be fixed in the body and rotating with it. At a given instant let  $X_a, Y_a, X_b, Y_b$  be the components parallel to the axes of the reactions of the supports *due to rotation only*. Let  $dM$  be the mass of an element of the rotor located at  $(x, y, z)$  distant  $r$  from the axis of rotation, and let the angle between  $x$  and  $r$  be  $\theta$ . Since the angular velocity is assumed to be constant, the only effective force acting on  $dM$  is  $r\omega^2 dM$  directed toward the axis. The system of reversed effective forces at the particles and the reactions of the supports due to rotation only will form a system in equilibrium. The conditions of this equilibrium may be expressed by the six equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0, \quad L_x = 0, \quad L_y = 0, \quad L_z = 0.$$

The third and sixth of these equations are clearly satisfied, since the forces are all perpendicular to and intersect  $OZ$ . The four remaining equations are

$$\begin{aligned} X_a + X_b + \int r\omega^2 dM \cos \theta &= 0, \\ Y_a + Y_b + \int r\omega^2 dM \sin \theta &= 0, \\ aY_a - bY_b + \int r\omega^2 dM \sin \theta z &= 0, \\ aX_a - bX_b + \int r\omega^2 dM \cos \theta z &= 0. \end{aligned}$$

Since  $r \cos \theta = x, r \sin \theta = y$ , we may write these equations in the form,

$$\begin{aligned} (1) \quad X_a + X_b + \omega^2 \bar{x}M &= 0, \\ (2) \quad Y_a + Y_b + \omega^2 \bar{y}M &= 0, \\ (3) \quad aY_a - bY_b + \omega^2 \int yz dM &= 0, \\ (4) \quad aX_a - bX_b + \omega^2 \int xz dM &= 0. \end{aligned}$$

where  $M$  is the mass of the rotor and  $\bar{x}$  and  $\bar{y}$  are the coordinates of its centroid. In order that the rotor be perfectly balanced when running, the reactions due to rotation only must all vanish. From the above equations we see that the conditions for this are

$$(5) \quad \bar{x} = 0, \quad \bar{y} = 0, \quad \int yz dM = 0, \quad \int xz dM = 0.$$



It may be noticed that the first two of these equations express the condition that the shaft would have standing balance, for if the centroid is on the axis of rotation there would not be any external torque tending to turn the shaft in smooth bearings when placed in any position at rest.

Suppose, however, that there is not perfect balance when the shaft is running, and that it is desired to balance the shaft by the addition of one or more bodies which are symmetric with respect to planes perpendicular to the axis of rotation; for example, by one or more circular disks whose plane faces are perpendicular to the axis of rotation.

For a body which is symmetric with respect to a plane perpendicular to the  $z$  axis we have, by § 155,

$$\int xzdm = \bar{x}\bar{z}M, \quad \int yzdm = \bar{y}\bar{z}M.$$

Let us see if the original rotor can be balanced by the addition of one such disk. If we assume that a disk of mass  $m_1$  be attached to the rotor with its center at  $(x_1, y_1, z_1)$  so that the rotor will be balanced for running, we must have

$$(6) \quad m_1x_1 + \bar{x}M = 0,$$

$$(7) \quad m_1y_1 + \bar{y}M = 0,$$

$$(8) \quad m_1y_1z_1 + \int yzdm = 0,$$

$$(9) \quad m_1x_1z_1 + \int xzdm = 0,$$

in which  $\bar{x}M$ ,  $\bar{y}M$ ,  $\int yzdm$ , and  $\int xzdm$  refer to the original rotor. Here there appear to be four unknowns,  $m_1$ ,  $x_1$ ,  $y_1$ ,  $z_1$ , to satisfy four equations; but  $m_1$  occurs only in the combinations  $m_1x_1$  and  $m_1y_1$ , and hence there are only three independent quantities,  $m_1x_1$ ,  $m_1y_1$ ,  $z_1$ . These three quantities cannot be chosen to satisfy the four (in general) independent equations. Hence, in general, an unbalanced shaft cannot be balanced for running by the addition of a single disk.

If we assume that two disks of masses  $m_1$  and  $m_2$  are attached to the shaft with their centers at  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  respectively, we have for the conditions that the shaft will then

be balanced,

$$(10) \quad m_1x_1 + m_2x_2 + \bar{x}M = 0,$$

$$(11) \quad m_1y_1 + m_2y_2 + \bar{y}M = 0,$$

$$(12) \quad m_1y_1z_1 + m_2y_2z_2 + \int yz dM = 0,$$

$$(13) \quad m_1x_1z_1 + m_2x_2z_2 + \int xz dM = 0.$$

Here are six quantities,  $m_1x_1$ ,  $m_2x_2$ ,  $m_1y_1$ ,  $m_2y_2$ ,  $z_1$ ,  $z_2$ , and these can evidently be determined so that the conditions for balance are satisfied. Moreover, since there are two more quantities than there are equations, we can, in general, choose two of the quantities at will, say  $z_1$  and  $z_2$ . It must be noticed, however, that  $z_1$  and  $z_2$  cannot be given the same value; the equations then become the same as those for a single weight, except that  $m_1x_1$  is replaced by  $m_1x_1 + m_2x_2$ , etc. It is therefore possible to balance the shaft by two disks with centers in two arbitrarily selected (different) planes perpendicular to the axis of rotation. Moreover, the masses of these disks can be arbitrarily given any convenient values.

It may be noticed that the conditions for balance, equations (5), or equations (10)–(13), do not include the distances from the origin to the bearings. From this it follows (a) that equations (5) must be satisfied, no matter where the origin may be chosen on the axis of rotation, (b) that the conditions of equilibrium are the same, no matter where the bearings are.

In order to determine the values of  $m_1x_1$ ,  $m_1y_1$ ,  $m_2x_2$ ,  $m_2y_2$  that satisfy equations (10)–(13) it is necessary to know the value of  $M\bar{x}$ ,  $M\bar{y}$ ,  $\int yz dM$ , and  $\int xz dM$ . If the rotor is made up of parts each of which is symmetric with respect to a plane perpendicular to the axis of rotation, with mass and position of centroid known, these quantities can be computed and the positions of the disks for balancing can be determined. In any case, if some means were found of accurately measuring the values of  $X_a$ ,  $Y_a$ ,  $X_b$ , and  $Y_b$  for a known speed of rotation, the values of  $M\bar{x}$ ,  $M\bar{y}$ ,  $\int yz dM$ , and  $\int xz dM$  could then be found from equations (1)–(4) and these values could then be substituted in equations (10)–(13).

The desired quantities for the balancing of the rotor could then be determined from equations (10)–(13).

In actual practice it is necessary to have a high-speed rotor balanced with great exactness, for at high speed a small amount of unbalance, which may be due to slight errors in manufacture, may cause serious trouble at the bearings.

Several machines have been devised for balancing high-speed rotors. Some of these machines and the methods of using them are described in Timoshenko's *Vibration Problems in Engineering*. It is not within the province of this book to discuss these machines and methods.

**160. Illustration.** As an application of the equations found for the determination of balance weights, let us consider what weights will be needed to balance a shaft with three equal cranks placed at intervals of  $120^\circ$  around the shaft and at equal distances apart along the shaft, Fig. 346.

Let the mass of each crank be  $m$  and let the distance from the axis to the centroid of each crank be  $d$ . Suppose it is desired to balance the shaft by two weights in planes distant  $a$  respectively from the centers of the end cranks, where  $a$  is the distance between the centers of adjacent cranks, measured along the axis of rotation. Let the origin be taken in one of the planes which is to contain the center of one of the

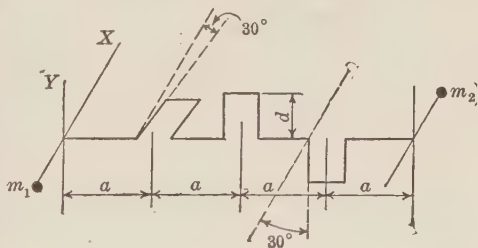


FIG. 346

balancing weights and choose the  $yz$  plane through the center of the middle crank. It is assumed that each crank is symmetric with respect to the plane through its center perpendicular to the axis of rotation. With this choice of axes we then have  $z_1 = 0$ ,  $z_2 = 4a$ . Equations (10) – (13) are then reduced to the following:

$$\begin{aligned}
 (1) \quad & m_1 x_1 + m_2 x_2 = 0, \\
 (2) \quad & m_1 y_1 + m_2 y_2 = 0, \\
 (3) \quad & m_2 y_2 (4a) + (-0.5d)am + d(2a)m + (-0.5d)(3a)m = 0, \\
 (4) \quad & m_2 x_2 (4a) + \left(\frac{d\sqrt{3}}{2}\right)am + 0 + \left(-\frac{d\sqrt{3}}{2}\right)(3a)m = 0.
 \end{aligned}$$

From (3) and (2) we find

$$m_2 y_2 = 0, \quad m_1 y_1 = 0.$$

From (4) and (1) we find

$$m_2 x_2 = \frac{\sqrt{3}}{4} d m, \quad m_1 x_1 = -\frac{\sqrt{3}}{4} d m.$$

The balancing weights are then to be placed in the  $xz$  plane as indicated in Fig. 346. If  $m_1$  and  $m_2$  are assigned arbitrary values, the values of  $x_1$  and  $x_2$  are then determined by the above equations.

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440. It is desired to balance the four weights, each of value  $w$ , placed on a shaft at distances  $d$  from the axis of the shaft at equal distances along the shaft and at intervals of  $90^\circ$  around the shaft, as indicated in Fig. 347. Find the

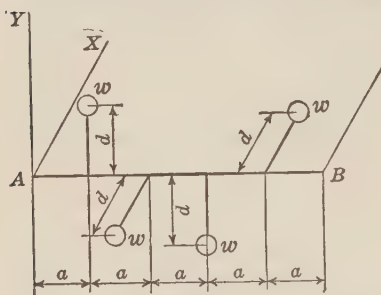


FIG. 347

positions of two weights, each equal to  $w$ , one in the plane through  $A$  and the other in the plane through  $B$ , that will balance the given weights for running.

441. Show that two weights, each symmetric with respect to a plane perpendicular to the axis of rotation and having their centroids in the same plane containing the axis of rotation, can be balanced by a single weight in that plane, and show how it would be located.

442. Show that two weights in the same plane perpendicular to the axis of rotation can be balanced for running by a single weight in that plane, and show how it would be located.

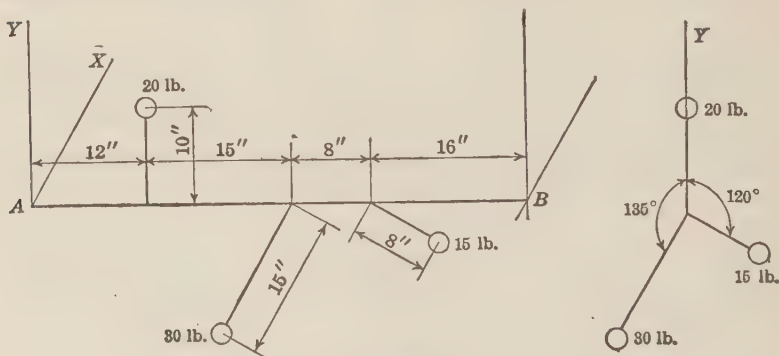


FIG. 348

443. In Fig. 348 the arms of the three weights are each perpendicular to the axis of rotation. Find where a weight of 10 lbs. should be attached to the shaft in the plane through  $A$  perpendicular to the shaft and where a weight of 8 lbs. should be attached in the plane through  $B$  perpendicular to the shaft so that the shaft will be balanced.

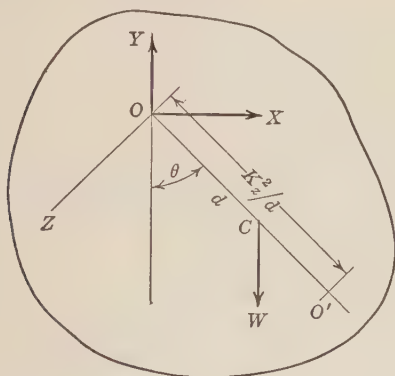


FIG. 349

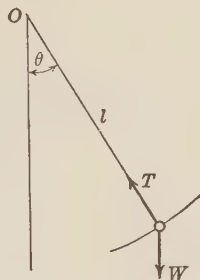


FIG. 350

**161. The Compound Pendulum.** A compound pendulum is defined as any body rotating about a fixed horizontal axis under the action of its own weight and the reaction of the axis of support.

Let a body be rotating about a fixed axis  $OZ$  and let Fig. 349 represent at any instant a section of the body which is perpendicular to  $OZ$  and which passes through the centroid  $C$  of the body. Let  $d$  be the distance of the centroid from the axis of rotation. The components of the reactions of the support are represented by  $X$  and  $Y$ . They are not necessarily in the plane section through the centroid.

If we consider moments from  $X$  toward  $Y$  to be positive and if we count angular acceleration  $\alpha$  in that direction, so that  $\alpha = d^2\theta/dt^2$ , we have by equation (6), § 152,

$$-Wd \sin \theta = I_z \frac{d^2\theta}{dt^2},$$

or

$$(1) \quad \frac{d^2\theta}{dt^2} = -\frac{gd}{k_z^2} \sin \theta,$$

where  $k_z$  is the radius of gyration of the body with respect to the axis of rotation.

If now we consider a simple pendulum, Fig. 350, we have for its motion

$$F_t = Ma_t,$$

that is,

$$-W \sin \theta = Ml\alpha = \frac{W}{g} l \frac{d^2\theta}{dt^2},$$

or

$$(2) \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta.$$

A comparison of equations (1) and (2) shows that the angular acceleration of a compound pendulum for a given value of  $\theta$  is precisely the same as that of a simple pendulum of length  $l$  for the same value of  $\theta$ , provided only that  $l = k_z^2/d$ . Hence if a compound pendulum and a simple pendulum of length  $l = k_z^2/d$  be started from rest with the same angular displacement at the same time they will vibrate at the same rate. The simple pendulum of length  $l = k_z^2/d$  is called the *equivalent simple pendulum*.

The period of oscillation of the compound pendulum is obtained at once from the period of oscillation of a simple pendulum by replacing  $l$  by  $k_z^2/d$ . Thus we get for the period of a compound pendulum for small oscillations

$$\tau = 2\pi \sqrt{\frac{k_z^2}{dg}}.$$

The point  $O$  of Fig. 349, in which the axis of rotation cuts the plane of motion, is called the center of suspension, and the point  $O'$  on  $OC$ , such that  $OO' = k_z^2/d$ , is called the *center of oscillation*.

If  $\bar{k}$  is the radius of gyration of the body with respect to an axis through the centroid parallel to the axis of rotation, we may write

$$k_z^2 = \bar{k}^2 + d^2;$$

therefore

$$l = \frac{\bar{k}^2}{d} + d;$$

and

$$\bar{k}^2 = d(l - d).$$



Suppose now that the body be set vibrating about the line of the body through  $O'$  parallel to the axis of rotation through  $O$ . The point  $O'$  then becomes the center of suspension. If  $l'$  denotes the length of the equivalent simple pendulum, we shall have

$$l' = \frac{\bar{k}^2}{CO'} + CO' = \frac{\bar{k}^2}{l-d} + l-d = d + l-d = l.$$

The pendulum would then vibrate in the same time as when suspended about the axis through  $O$ . The centers of suspension and oscillation are thus said to be interchangeable.

The formula for the period of vibration,

$$\tau = 2\pi \sqrt{\frac{\bar{k}_z^2}{gd}}$$

provides a means of computing experimentally the moment of inertia of any body with respect to any axis. The body is suspended about the desired axis and set in small oscillations. The period is then found by observation and the value of  $k_z$  may be computed from the formula for the period of vibration. From this value of  $k_z$  and the weight of the body the moment of inertia may be computed.

**162. The Torsion Balance.** A torsion balance consists of a body suspended by a slender rod, or wire, attached rigidly to the body and to a fixed support. The centroid of the body lies in the center line of the wire. The body oscillates about the wire as an axis due to the torque exerted by the twisted wire. The suspending wire is assumed to be sufficiently slender so that its moment of inertia may be neglected, and it is made of material which may be assumed to be perfectly elastic without serious error.

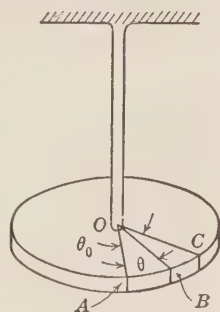


FIG. 351

Let  $OA$  (Fig. 351) be the position of a radial line of the body, perpendicular to the axis of rotation, when the body is at rest. Let the body be turned through an angle  $\theta_0$  and then released. Since the wire is twisted, it exerts a torque on the body, which

tends to restore it to the neutral position that it occupies when at rest. Experiment shows that this torque is proportional to the angular displacement,  $\theta$ , from the neutral position. Hence, by equation (6), § 152, we have for the motion of the disk, starting from the position where  $\theta = \theta_0$ ,

$$-C\theta = I\alpha,$$

the minus sign being used since  $\alpha$  is to be taken as  $d^2\theta/dt^2$ , which is opposite in sign to the torque.

Replacing  $\alpha$  by  $\omega d\omega/d\theta$ , we have

$$\int_0^\omega I\omega d\omega = -C \int_{\theta_0}^\theta \theta d\theta,$$

or

$$(1) \quad I\omega^2 = C(\theta_0^2 - \theta^2).$$

This equation shows that the angular velocity of the body is the same for any negative value of  $\theta$  as for the numerically equal positive value, and that it is the same whether  $\theta$  is increasing or decreasing. The motion is therefore periodic, the body vibrating through the angle  $\theta_0$  on both sides of the neutral position.

Considering the motion from  $\theta = 0$  to  $\theta = \theta_0$ ,  $\omega = d\theta/dt$  is positive and we have

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = \sqrt{\frac{C}{I}} \int_0^{\tau/4} dt,$$

where  $\tau$  is the period of vibration. From this equation we find

$$\tau = 2\pi\sqrt{I/C}.$$

In this formula  $C$  is a constant depending upon the length, the size, and the rigidity of the wire. In any given case the value of  $C$  may be determined by measuring the angle turned through when a given torque is applied. Thus, if a torque  $L_1$  ft.-lbs. applied to the wire twists it through  $\theta_1$  radians, the value of  $C$  is  $L_1/\theta_1$ .

### PROBLEMS

444. A slender rod of length  $l$  is suspended from a horizontal axis through one end and vibrates in a plane perpendicular to this axis. Find the time of a small oscillation and show that the length of the equivalent simple pendulum is  $2l/3$ .

445. Show that the period of oscillation of a uniform slender rod about an axis perpendicular to its length is least when the distance from the center of suspension to the center is  $\sqrt{3}/6$  times the length of the rod; and show also that the center of oscillation is then at the same distance from the center of the rod.

446. Show that the period of oscillation of a uniform circular disk about an axis perpendicular to its face is least when the distance from the axis to the center of the disk is  $\sqrt{2}/2$  times the radius.

447. Let  $C$  be the centroid of a body. In a plane containing the centroid, two circles of radii  $r$  and  $\bar{k}^2/r$  are drawn, with centers at  $C$ , where  $\bar{k}$  is the radius of gyration of the body with respect to the line through  $C$  perpendicular to the given plane. Show that the period of oscillation will be the same for all axes perpendicular to the given plane which intersect the circumference of either circle.

Show also that the period of oscillation for all axes perpendicular to the plane is least when the distance of the axis from  $C$  is  $\bar{k}$ , and that the length of the equivalent simple pendulum is then  $2\bar{k}$ .

448. A torsion balance is composed of a wire to which are attached two circular disks rigidly connected as shown in Fig. 352 by four rods. Each disk is 8 in. in diameter and weighs 0.87 lb. Each of the rods is 6 in. long between the disks, 0.25 in. in diameter and weighs 0.082 lb., and its center line is 3 in. from the center line of the disks. When the wire is twisted and then released, it is found that 20 oscillations are made in a minute. A body is then placed in the cage with its centroid in the line of the supporting wire, and it is then found that 20 oscillations are made in 2 min. 30 sec. Find the moment of inertia of the body about the vertical axis through the centroid in the given position.

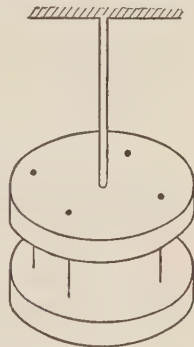


FIG. 352

**163. Motion of the Centroid of a Body.** If  $a_x, a_y, a_z$  are the components of the acceleration of any element of mass  $dM$  of a body having any motion, the effective forces acting on the particle parallel to the coordinate axes are  $a_x dM, a_y dM, a_z dM$ . By D'Alembert's principle, the impressed forces and the effective forces acting on the body may be equated. Thus, if  $\Sigma X$  is the sum, parallel to the  $x$  axis, of the components of the impressed forces acting on the body, we have

$$\Sigma X = \Sigma a_x dM.$$

If  $\bar{x}$  is the abscissa of the centroid of the body,

$$M\bar{x} = \Sigma x dM.$$

Differentiating both sides of this equation, we have

$$M \frac{d\bar{x}}{dt} = \Sigma \frac{dx}{dt} dM,$$

and

$$M \frac{d^2\bar{x}}{dt^2} = \Sigma \frac{d^2x}{dt^2} dM = \Sigma a_x dM.$$

Hence, if  $\bar{a}_x$  is the component, parallel to the  $x$  axis, of the acceleration of the centroid of the body, we may write

$$\Sigma X = M\bar{a}_x,$$

and, similarly

$$\Sigma Y = M\bar{a}_y,$$

$$\Sigma Z = M\bar{a}_z.$$

Hence, in any motion of a body, the acceleration of the centroid of the body is the same as if the whole mass were concentrated at the centroid and were acted upon by forces parallel and equal to all of the impressed forces that act on the body.

**164. Impulsive Forces.** An *impulsive force* is one which is suddenly applied and that lasts for a very brief time, usually for a very small fractional part of a second.

When two bodies *collide*, or come together in *impact*, they exert impulsive forces on each other.

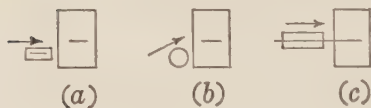


FIG. 353

The impact is said to be *direct* if the relative velocity of the striking surfaces is in the

direction of the normal to those surfaces. Impact that is not direct is said to be *oblique*.

When the normal to the surfaces at the point of contact (or center of the forces exerted by one body on the other) passes through the centroids of both bodies, the impact is said to be *central*. Thus, in Fig. 353, (a) represents direct excentric impact, (b) represents oblique impact, and (c) represents direct central impact.

Experience indicates that when two bodies come together in impact, each undergoes a deformation and then, in general, if the impact is not too severe, a partial or complete restoration of form. Bodies in which there is no restoration of form after deformation are said to be *inelastic*. Those in which there is complete restoration of form are said to be *perfectly elastic*. Those in which there is partial restoration of form are said to be *partially elastic*.

The time of impact is, in general, very short—of the order of  $10^{-3}$  sec., or  $10^{-4}$  sec. It is customary to divide this time into two periods, the *period of deformation* and the *period of restitution*. The period of deformation is that portion of the time of impact during which the bodies are being compressed. The period of restitution is the remaining portion of the time of impact if the bodies are not inelastic. In the case of elastic bodies the pressure between the surfaces in contact increases from zero to a maximum during the period of deformation and then decreases again to zero during the period of restitution.

At the end of the period of deformation the components of the velocities of the two surfaces in contact, normal to these surfaces, are equal, but at all other times during the impact these components of velocity are unequal.

**165. Direct Central Impact.** As an example of direct central impact consider two spheres which collide while moving along the straight line which joins their centers. Let the masses of the spheres be  $m_1$  and  $m_2$  and let the velocities be  $v_1$  and  $v_2$  respectively at the instant that they come in contact, Fig. 354 (a). During

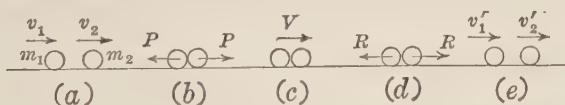


FIG. 354

the period of deformation there is a force  $P$  acting between the bodies, which causes an acceleration of one of the bodies,  $m_2$ , and a retardation of the other,  $m_1$ . Counting the direction of  $v_1$  as positive, we have for the motion of the two bodies,

$$-P = m_1 \frac{dv}{dt}, \quad P = m_2 \frac{dv}{dt}.$$



If  $t_1$  is the time of deformation and if we assume that at the end of this period all particles of the two bodies have a common velocity  $V$ , we may write

$$(1) \quad - \int_0^{t_1} P dt = m_1 \int_{v_1}^V dv = m_1(V - v_1),$$

$$(2) \quad \int_0^{t_1} P dt = m_2 \int_{v_2}^V dv = m_2(V - v_2).$$

If we add these equations we obtain

$$0 = m_1(V - v_1) + m_2(V - v_2).$$

From this equation we have

$$(3) \quad m_1 v_1 + m_2 v_2 = (m_1 + m_2)V.$$

This equation states that the sum of the momenta of the bodies at the end of the period of deformation is the same as at the beginning of the period. Indeed, it is evident that this sum remains unchanged throughout the impact; for the change in momentum of  $m_1$  in any time  $t'$  measured from the beginning of the impact is

$$m_1(v_1' - v_1) = - \int_0^{t'} P dt,$$

while that for  $m_2$  is

$$m_2(v_2' - v_2) = \int_0^{t'} P dt.$$

The total change in momentum of the two bodies is, therefore,

$$m_1(v_1' - v_1) + m_2(v_2' - v_2) = - \int_0^{t'} P dt + \int_0^{t'} P dt = 0.$$

Next consider the period of restitution. In order to distinguish between the two periods, let the force acting between the bodies during the period of restitution be denoted by  $R$ . We then have for the motion of the two bodies during the period of restitution

$$-R = m_1 \frac{dv}{dt}, \quad R = m_2 \frac{dv}{dt}.$$



Let us assume that the period of restitution extends from  $t_1$  to  $t_2$  and that the velocities of  $m_1$  and  $m_2$  at the end of the period are  $v_1'$  and  $v_2'$  respectively. We shall then have

$$(4) \quad - \int_{t_1}^{t_2} R dt = m_1(v_1' - V),$$

$$(5) \quad \int_{t_1}^{t_2} R dt = m_2(v_2' - V).$$

The value of  $\int_{t_1}^{t_2} R dt$  is not, in general, the same as  $\int_0^{t_1} P dt$ . However, experiments show that for any two materials in impact the ratio of  $\int_{t_1}^{t_2} R dt$  to  $\int_0^{t_1} P dt$  is, for a limited range of velocities, nearly constant. It was Newton's belief, based upon experiments, that this ratio is constant for any two materials; later experiments show that it is not absolutely constant but that it decreases with the velocity of impact. Newton found that for a steel ball the ratio was 5/9, for ivory it was 8/9, and for glass it was 15/16.

The ratio of  $\int_{t_1}^{t_2} R dt$  to  $\int_0^{t_1} P dt$  is called the *coefficient of restitution* and is represented by the letter  $e$ ; hence we may write

$$(6) \quad e = \frac{\int_{t_1}^{t_2} R dt}{\int_0^{t_1} P dt}.$$

In case the restoration of form is complete during the period of restitution,  $R$  takes in inverse order all of the values assumed by  $P$  during the first period.

The quantity  $P dt$  is called the impulse of the force  $P$  during the time  $dt$ , and  $\int_0^{t_1} P dt$  is then the impulse of the force  $P$  during the time  $t_1$ . It follows from this definition that the coefficient of restitution is the ratio of the impulse of the force of impact during the period of restitution to the impulse of the force of impact during the period of deformation. If the two bodies are perfectly elastic, the value of  $e$  is unity. If  $e = 0$ , the impact is inelastic.

Assuming that  $e$  is a known quantity, the value of  $v_1'$  may be found by the combination of equations (1), (4), and (3). Similarly, the value of  $v_2'$  may be found by the combination of equations (2), (5), and (3).

**166. Oblique Impact.** As an illustration of oblique impact, assume that two smooth spheres, each of mass  $m$ , for which  $e = 0.9$ , collide as shown in Fig. 355.

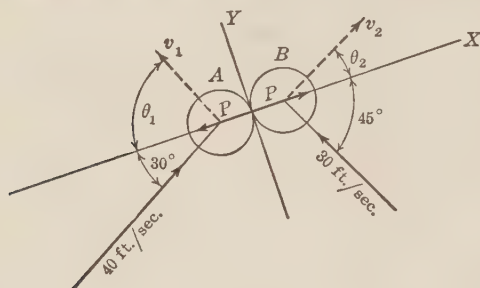


FIG. 355

Choose a set of rectangular axes, letting the  $y$  axis be the common tangent to the two spheres at the instant of contact and the  $x$  axis be the line through the centers at

that instant. We may assume the force acting between the spheres to be along the line of centers. For the period of deformation we may write the following equations respectively for the spheres  $A$  and  $B$ ,

$$(1) \quad - \int_0^{t_1} P dt = m \int_{34.64}^{V_x} dv = m(V_x - 34.64),$$

$$(2) \quad \int_0^{t_1} P dt = m \int_{-21.21}^{V_x} dv = m(V_x + 21.21),$$

where  $V_x$  denotes the common component of the velocity of the spheres along the line of centers. Adding these equations and solving for  $V_x$ , we find

$$(3) \quad V_x = 6.715 \text{ ft./sec.}$$

For the period of restitution we have

$$(4) \quad - \int_{t_1}^{t_2} R dt = m(v_x' - V_x),$$

$$(5) \quad \int_{t_1}^{t_2} R dt = m(v_x'' - V_x),$$

where  $v_x'$  and  $v_x''$  represent respectively the  $x$  components of the velocities of  $A$  and  $B$  at the end of impact. From equations (4), (1), and (3) we obtain

$$e = 0.9 = \frac{v_x' - V_x}{V_x - 34.64} = \frac{v_x' - 6.715}{-27.925},$$

from which we find

$$v_x' = -18.42 \text{ ft./sec.}$$

Similarly, we find

$$v_x'' = 31.85 \text{ ft./sec.}$$

Since there has been no force acting on either body in the direction of the  $y$  axis, no change in velocity will have occurred in that direction. The sphere  $A$  will then have at the end of impact a velocity

$$v_1 = \{(18.42)^2 + (20)^2\}^{1/2} = 27.20 \text{ ft./sec.},$$

at an angle

$$\theta_1 = \tan^{-1} \frac{20}{18.42} = 47^\circ 21'.$$

with the negative  $x$  axis.

The sphere  $B$  will have at the end of impact a velocity

$$v_2 = \{(31.85)^2 + (21.21)^2\}^{1/2} = 38.26 \text{ ft./sec.}$$

at an angle

$$\theta_2 = \tan^{-1} \frac{21.21}{31.85} = 33^\circ 40'$$

with the positive  $x$  axis.

It is interesting to find the kinetic energy of the spheres before and after the impact. These are respectively

$$E_1 = \frac{1}{2}m(40)^2 + \frac{1}{2}m(30)^2 = 1250m \text{ ft.-lbs.},$$

$$E_2 = \frac{1}{2}m(27.20)^2 + \frac{1}{2}m(38.26)^2 = 1101.8m \text{ ft.-lbs.}$$

Hence there has been a loss of  $148.2m$  ft.-lbs., which is 11.85 per cent of the original kinetic energy.

As in the case of direct central impact, there has been no

change in the total momentum of the two bodies, provided that momentum be regarded as a vector quantity. Momentum is so defined, the direction of the momentum vector being the same as the direction of the velocity vector. It is not true that the sum of the numerical values of  $m_1v_1$  and  $m_2v_2$  before oblique impact is the same as the corresponding sum after impact. In order to prove that there is no vector change in momentum in oblique impact, it is only necessary to note that there is no change in the component of momentum in the direction of either of the coordinate axes.

**167. Direct Excentric Impact.** EXAMPLE 1. Let Fig. 356 represent a rectangular slab of weight 60 lbs. which is suspended from a smooth horizontal pin at  $O$ . The slab is raised to a horizontal position and released and after rotating through  $90^\circ$  strikes a 10-lb. ball as shown in the figure. If the coefficient of restitution is 0.55, what will be the velocity of the ball and the angular velocity of the slab after impact?

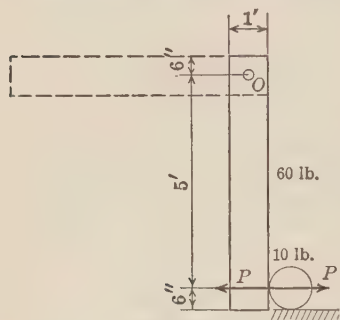


FIG. 356

It is necessary to find the angular velocity with which the slab strikes the ball. For this purpose it is convenient to use the work equation. We have

$$\frac{1}{2} I_0 \omega_0^2 = Wh = (60)(2.5) = 150 \text{ ft.-lbs.}$$

$$I_0 = \frac{60}{g} \frac{1}{12} (36 + 1) + \frac{60}{g} \left( \frac{5}{2} \right)^2 = \frac{560}{g}.$$

Therefore

$$\omega_0^2 = \frac{15g}{28} = 17.25,$$

$$\omega_0 = 4.153 \text{ rad./sec.}$$

During the period of deformation, a period of time so short that no appreciable change in the positions of the bodies occurs, we have the following equations. For the slab

$$-5P = I_0 \alpha = \frac{560}{g} \frac{d\omega}{dt}.$$

For the ball

$$P = Ma = \frac{10}{g} \frac{dv}{dt}.$$

From these equations we obtain

$$(1) \quad -5 \int_0^{v_1} P dt = \frac{560}{g} \int_{\omega_0}^{\omega_1} d\omega = \frac{560}{g} (\omega_1 - \omega_0),$$

$$(2) \quad \int_0^{v_1} P dt = \frac{10}{g} \int_0^{v_1} dv = \frac{10}{g} v_1,$$

where  $v_1$  is the velocity of the ball at the end of the period of deformation and  $\omega_1$  is the angular velocity of the slab at that instant. Also at the end of the period of deformation the point in the axis of the slab which is in the line of the velocity of the center of the ball has the same velocity as the center of the ball. Therefore

$$(3) \quad v_1 = 5\omega_1.$$

Solving equations (1), (2), and (3), we find

$$\omega_1 = 2.871 \text{ rad./sec.}, \quad v_1 = 14.355 \text{ ft./sec.}$$

For the period of restitution we have

$$(4) \quad -5 \int_{v_1}^v R dt = \frac{560}{g} \int_{\omega_1}^{\omega} d\omega = \frac{560}{g} (\omega - \omega_1),$$

$$(5) \quad \int_{v_1}^v R dt = \frac{10}{g} \int_{v_1}^v dv = \frac{10}{g} (v - v_1),$$

where  $\omega$  is the angular velocity of the slab at the end of the impact and  $v$  is the velocity of the ball at that instant.

Dividing the members of (4) by the corresponding members of (5), we have

$$e = 0.55 = \frac{\omega - \omega_1}{\omega_1 - \omega_0} = \frac{\omega - 2.871}{-1.282},$$

from which we find

$$\omega = 2.166 \text{ rad./sec.}$$

Treating (5) and (2) similarly, we have

$$e = 0.55 = \frac{v - v_1}{v_1} = \frac{v - 14.355}{14.355},$$

from which we find

$$v = 22.25 \text{ ft./sec.}$$

**EXAMPLE 2.** Using the same bodies as in Example 1, assume that the slab is lying at rest on a smooth horizontal surface, but not pivoted to this surface, when it is struck by the ball moving 40 ft./sec. in a horizontal direction perpendicular to the 6 ft. edge of the slab, as shown in Fig. 357. It is desired to find the velocity of the ball and to determine the motion of the slab at the end of impact, given  $e = 0.55$ .

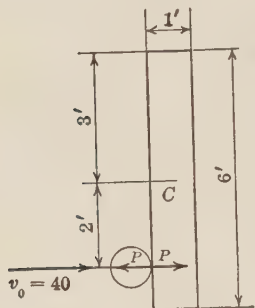


FIG. 357

Here we use the theorem that the centroid of a body moves just as if the mass of the body were concentrated at the centroid and all of the impressed forces were acting there (§ 163), and the theorem that in the case of plane motion of a body the torque of the impressed forces about the centroidal axis perpendicular to the plane of motion is equal to the product of the moment of inertia of the body about that axis and the angular acceleration of the body (§ 157). We then have two equations of motion for the slab and one equation for the ball during each period. If we count velocity to the right as positive and angular velocity counterclockwise as positive, we then have during the period of deformation, for the slab,

$$P = \frac{60}{g} \frac{dv}{dt}, \quad 2P = \bar{I} \frac{d\omega}{dt} = \frac{185}{g} \frac{d\omega}{dt};$$

and for the ball,

$$-P = \frac{10}{g} \frac{dv}{dt}.$$

If we let  $v_1$  be the velocity of the ball at the end of the period of deformation,  $\omega_1$  the angular velocity of the slab, and  $v_1'$  the velocity of the centroid of the slab at that instant, by integration of the above equations, we obtain

$$(1) \quad \int_0^{t_1} P dt = \frac{60}{g} \int_0^{v_1'} dv = \frac{60}{g} v_1',$$

$$(2) \quad 2 \int_0^{t_1} P dt = \frac{185}{g} \int_0^{\omega_1} d\omega = \frac{185}{g} \omega_1,$$

$$(3) \quad - \int_0^{t_1} P dt = \frac{10}{g} \int_{40}^{v_1} dv = \frac{10}{g} (v_1 - 40).$$



Also at the end of the period of deformation the velocity of the ball is the same as the velocity of the point in the center line of the rod that lies in the line of the velocity of the ball. The velocity of this point is made up of the velocity of the centroid  $C$  and its velocity relative to  $C$ , or  $v_1' + 2\omega_1$ . We therefore have

$$(4) \quad v_1 = v_1' + 2\omega_1.$$

Solving equations (1) - (4), we find

$$v_1' = 4.821 \text{ ft./sec.}, \quad \omega_1 = 3.127 \text{ rad./sec.}, \quad v_1 = 11.075 \text{ ft./sec.}$$

For the period of restitution the equations are

$$(5) \quad \int_{t_1}^{t_2} R dt = \frac{60}{g} (v' - v_1'),$$

$$(6) \quad 2 \int_{t_1}^{t_2} R dt = \frac{185}{g} (\omega - \omega_1),$$

$$(7) \quad - \int_{t_1}^{t_2} R dt = \frac{10}{g} (v - v_1),$$

where  $v'$ ,  $\omega$ , and  $v$  are respectively the velocity of the centroid of the slab, the angular velocity of the slab, and the velocity of the ball at the end of impact. In addition to these equations we have

$$(8) \quad e = \frac{\int_{t_1}^{t_2} R dt}{\int_0^{t_1} P dt}.$$

Solving equations (1), (5), and (8), we find the value of  $v'$ . Similarly, from (2), (6), and (8) we obtain the value of  $\omega$ , and from (3), (7), and (8) the value of  $v$ . The results are

$$v' = 7.473 \text{ ft./sec.}, \quad \omega = 4.847 \text{ rad./sec.}, \quad v = -4.834 \text{ ft./sec.}$$

Computing the kinetic energy before and after impact, we have, before impact

$$E_1 = \frac{1}{2} \frac{10}{g} (40)^2 = \frac{8000}{g} \text{ ft.-lbs.},$$

after impact

$$E_2 = \frac{1}{2} \frac{10}{g} (4.834)^2 + \frac{1}{2} \frac{185}{g} (4.847)^2 + \frac{1}{2} \frac{60}{g} (7.473)^2 = \frac{3965}{g} \text{ ft.-lbs.}$$

Hence during the impact there is a loss of 50.44 percent of the original kinetic energy.

## PROBLEMS

449. Show that if two perfectly elastic balls of equal mass collide in direct central impact they exchange velocities. If one of the balls is at rest when struck by the other, what happens?

450. If several perfectly elastic balls of equal mass were at rest in a straight row on a smooth horizontal surface, just touching each other (Fig. 358), and

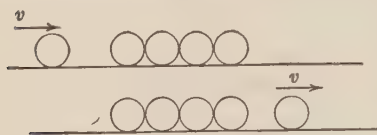


FIG. 358

were struck by another ball of the same mass and material moving in the line of the balls, show that the striking ball would be brought to rest where it strikes and the last ball at the other end would take its velocity, all other balls remaining at rest.

451. Show that if two of the set of equal balls, moving together, strike the row of balls in the preceding problem, then two balls at the other end of the row will move off with the velocity of the striking balls and all others will remain at rest.

452. Show that if a single ball of double the mass of each ball in the row of elastic balls of Problem 450 strikes the line with velocity  $v$ , the first ball at the other end of the row will move off with velocity  $4v/3$ , the next to it will take a velocity  $4v/9$ , the third will take a velocity  $4v/27$ , etc. until all balls including the striking ball are in motion. If there are four balls standing in the row, with what velocity will the striking ball move on after the impact?

453. Show that if the striking ball of Problem 450 have a mass equal to one-half that of each of the balls in the row, it will be reversed in direction with a velocity equal to one-third of its original velocity, while the last ball at the other end of the row will move off with a velocity equal to two-thirds of the velocity of the striking ball, all other balls of the row remaining at rest.

454. Prove that if two inelastic balls moving in opposite directions with velocities inversely proportional to their masses, collide, both will be brought to rest.

455. Prove that if a ball falls from a height  $H$  upon a very large mass with a horizontal surface and rebounds to a height  $h$ , the coefficient of restitution of the materials of the two bodies is

$$e = \sqrt{\frac{h}{H}}.$$

456. A bullet weighing 1 oz. is fired horizontally into a box of sand (inelastic) and remains imbedded in it. The box is suspended by strings attached to a horizontal support which is 4 ft. above the center of the box of sand, Fig. 359. The impact of the bullet causes the box to swing through  $42^\circ$ . Find (a) the velocity of impact of the bullet, (b) the percentage of the kinetic energy lost in impact, (c) the maximum tension in the strings.

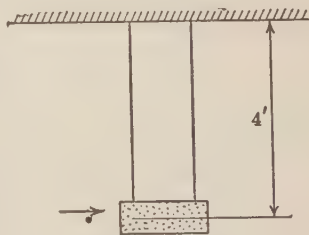


FIG. 359

457. Prove that if two bodies with masses  $m_1$  and  $m_2$ , moving with initial velocities  $v_1$  and  $v_2$  respectively, have direct central impact, the loss in kinetic energy in impact is

$$\frac{m_1 m_2 (1 - e^2)}{2(m_1 + m_2)} (v_1 - v_2)^2.$$

What is the loss in kinetic energy if the bodies are perfectly elastic?

(It must be noted that in this formula  $v_1$  and  $v_2$  are directed quantities, so that if the bodies are moving in opposite directions before impact, one of the velocities must be counted as negative).

458. A rod of length 5 ft. and weight 20 lbs. is suspended from a horizontal axis which is 1 ft. from one end of the rod. A 5-lb. ball moving horizontally with velocity 40 ft./sec. strikes the rod at a point 1 ft. from the other end. If  $e = 0.5$ , find (a) the velocity of the ball after impact, (b) the angular velocity of the rod after impact, (c) the angle through which the rod will swing, (d) the percentage of kinetic energy lost in impact.

459. A square block of wood 2 ft. by 2 ft. by 4 in., weighing 50 lbs., is lying at rest on a smooth table with a large face in contact with the table. A ball weighing 5 lbs. moving horizontally with velocity 30 ft./sec., in a direction perpendicular to a face of the block, strikes the block at a point 2 in. from a corner. If  $e = 0.10$ , find after impact (a) the velocity of the ball, (b) the velocity of the center of the block, (c) the angular velocity of the block. What percentage of the kinetic energy is lost in impact?

460. Assuming that the impact is inelastic, show that if a pile-driver weighing  $W_1$  lbs. falls a distance  $h$  ft. and drives a pile weighing  $W_2$  lbs. into the ground a short distance  $s$  ft., the average resistance offered by the ground to the passage of the pile is approximately

$$R = \frac{W_1^2}{W_1 + W_2} \frac{h}{s} \text{ lbs.}$$

(SUGGESTION. Equate the work done by gravity on the falling weight to the work done against the resistance of the earth plus the kinetic energy lost in impact.)

461. Two equal spheres for which  $e = 0.90$  impinge as shown in Fig. 360. Find the velocity in magnitude and direction of each sphere after impact.

462. A sphere moving with velocity  $v$  strikes a smooth fixed surface at an angle  $\alpha$  with the normal to the surface. If  $e$  is the coefficient of restitution, find the velocity  $v'$  of the rebound of the sphere and the angle  $\beta$  which this velocity makes with the normal to the surface.

Under what condition will  $v' = v$  and  $\beta = \alpha$ ?

463. A ball is projected with velocity 50 ft./sec. against a smooth plane surface at an angle of  $40^\circ$  with the normal. It leaves at an angle of  $50^\circ$  with the normal. Find  $e$  and the velocity with which it leaves.

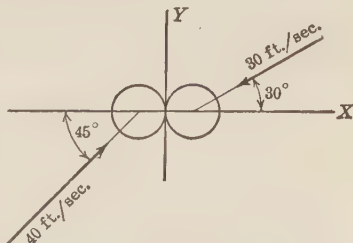


FIG. 360

**168. Center of Percussion and Center of Instantaneous Rotation.** Consider a body at rest to be struck a blow and to be free to move in a plane containing the centroid of the body and the

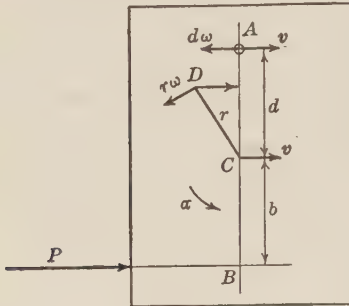


FIG. 361

line in which the force of the blow acts. For example, let a slab lying at rest on a smooth horizontal plane be struck a blow in a horizontal plane containing the centroid of the body, Fig. 361. For the very brief time in which the force of the blow is acting we may neglect any change in the position of the body. During the blow the ac-

celeration,  $dv/dt$ , of the centroid and the angular acceleration,  $d\omega/dt$ , are determined by the equations

$$P = M \frac{dv}{dt},$$

$$bP = M\bar{k}^2 \frac{d\omega}{dt},$$

where  $M$  is the mass of the body struck and  $\bar{k}$  is the radius of gyration of the body with respect to an axis through the centroid, perpendicular to the plane which contains the centroid and the line of the force  $P$  of the blow. If we let  $t$  be the very short time from the beginning of the blow to any other instant while the blow is acting, we have by integration of the above equations

$$\int_0^t P dt = M \int_0^v dv = Mv,$$

$$b \int_0^t P dt = M\bar{k}^2 \int_0^\omega d\omega = M\bar{k}^2\omega,$$

where  $v$  and  $\omega$  represent respectively the velocity of the centroid and the angular velocity of the body at the time  $t$ .

By division of the above equations, we obtain

$$\frac{1}{b} = \frac{v}{\bar{k}^2\omega},$$

or

$$v = \frac{\bar{k}^2}{b} \omega.$$

Since the centroid was at rest at the beginning of the blow, the velocity acquired in the time  $t$  will be in the direction of the force of the blow. Any other point, as  $D$ , distant  $r$  from  $C$ , will have at this instant a velocity made up of  $v$  and  $r\omega$ , perpendicular to  $r$ . It is therefore evident that if a point  $A$  of the body, on the line which passes through the centroid and is perpendicular to the line in which the force is acting, at a distance  $d$  from  $C$  such that

$$d = \frac{\bar{k}^2}{b},$$

has a velocity equal to zero. The body then starts to turn about an axis through  $A$  at the instant that the blow is struck. The point  $A$  is called the *instantaneous center* of rotation for the given blow. The point  $B$ , the foot of the perpendicular from the centroid to the line of the blow, is called the *center of percussion* for the given blow.

It should be noted that the center of percussion and the corresponding center of instantaneous rotation are related just as the center of oscillation and the center of suspension are related; that is, they lie on a line through the centroid, on different sides of the centroid, and at distances from the centroid whose product is equal to  $\bar{k}^2$ .

### PROBLEMS

464. A slender rod 3 ft. long, lying on a smooth table, is struck a blow at right angles to the length of the rod at a point 6 in. from one end of the rod. What point does the rod begin to turn about?

465. A slender rod of length  $l$  is suspended about a horizontal axis through one end. At what point should a horizontal blow be struck so that the supporting axis will not be subject to a shock?

466. At what point should a slender rod 3 ft. long be grasped so that when a blow is struck at one end of the rod the hand will not receive a shock?

467. A cylinder of radius  $r$  and altitude  $h$  is suspended from a horizontal axis along the diameter of one end. How far from this end should a horizontal blow be struck so that there will be no shock at the support? If the axis of support passed through the centroid of the body would it be possible to strike a blow that would not cause a shock at the support?



**169. Motion of a Rigid Body with One Point Fixed.** Let a body in motion have a fixed point,  $O$ , Fig. 362.

Since not all points of the body are at rest, a point  $A$  of the body may be chosen which at a given instant has a velocity  $v_a$ , not zero. Then  $v_a$  must be perpendicular to the line  $OA$ , for

otherwise the distance  $OA$  would be changing. Consider a plane section of the body containing  $OA$  and perpendicular to  $v_a$ . Not all points of the body in this plane can have zero velocity, for then the whole body would be at rest at the instant, which is contrary to the supposition. Let  $B$  be a point of the body in this plane, not on  $OA$ , whose velocity,  $v_b$ , is not zero. Then  $v_b$  is perpendicular to  $BA$  and to  $BO$  and

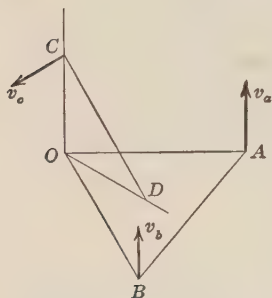


FIG. 362

hence to the plane  $OAB$ . That is, any point of the body in a plane through  $A$ , perpendicular to  $v_a$ , is moving perpendicular to that plane, if in motion at all.

Next, consider a point,  $C$ , of the body in a line through  $O$  perpendicular to the plane  $OAB$ . This point cannot be at rest, for then the distance  $CA$  would be changing, since  $CA$  is not perpendicular to  $v_a$ . Moreover,  $v_c$  must be perpendicular to  $OC$ . Then, just as with point  $A$ , we can say that any point of the body in a plane through  $C$ , perpendicular to  $v_c$ , must be moving perpendicular to that plane, if it is in motion at all. Therefore any point  $D$  of the body, in the intersection of the two planes  $OAB$  and  $OCD$ , must have zero velocity, for it cannot at the same time have a velocity perpendicular to each of two intersecting planes. The motion of the body at the given instant must therefore be one of instantaneous rotation about the line  $OD$ .

Hence, if a rigid body is in motion and has one fixed point, its motion must be one of instantaneous rotation about some line through that fixed point.

**170. Simplification of Any Motion of a Body.** Consider a body having any motion in space. Assume a set of fixed rectangular axes  $O - XYZ$  and let  $O_1 - X'Y'Z'$  be a set of moving



axes parallel to the fixed axes but with the point  $O_1$ , fixed relative to the body, as origin, Fig. 363. If the point  $O_1$ , referred to the fixed axes, is  $(x_1, y_1, z_1)$ , and if the coordinates of any point  $P$  of the body, referred to the fixed and moving axes respectively, are  $(x, y, z)$  and  $(x', y', z')$ , then  $x = x_1 + x', y = y_1 + y', z = z_1 + z'$ .

Hence the components of the velocity of  $P$  are

$$\frac{dx}{dt} = \frac{dx_1}{dt} + \frac{dx'}{dt},$$

$$\frac{dy}{dt} = \frac{dy_1}{dt} + \frac{dy'}{dt},$$

$$\frac{dz}{dt} = \frac{dz_1}{dt} + \frac{dz'}{dt}.$$

Now  $dx_1/dt, dy_1/dt, dz_1/dt$  are the components of the velocity of  $O_1$ , and  $dx'/dt, dy'/dt, dz'/dt$  are

the components of the velocity that  $P$  would have if  $O_1$  were at rest. This latter motion has been shown in § 169 to be one of rotation about a line through  $O_1$ .

Hence, in any motion of a body in space, the velocities of all points of the body are composed of the velocity of any selected point of the body and the velocity due to rotation about some axis through that point.

The angular velocity about the line through  $O_1$  may be represented vectorially and resolved into components  $\omega_x, \omega_y, \omega_z$  about the axes  $O_1X', O_1Y', O_1Z'$  respectively, and the components of velocity,  $dx'/dt, dy'/dt, dz'/dt$ , may then be expressed in terms of the components of the angular velocity and the coordinates of the point. Thus, in Fig. 363 the component parallel to  $O_1Z'$  of the velocity of  $P$  relative to the moving axes is

$$r_1\omega_x \cos \theta - r_2\omega_y \cos \phi$$

or

$$y'\omega_x - x'\omega_y.$$

The total component of the velocity of  $P$  parallel to the  $Z$  axis is,

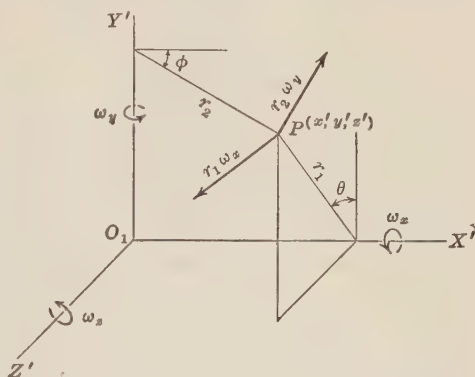


FIG. 363

therefore,

$$v_z = \frac{dz}{dt} = \frac{dz_1}{dt} + y'\omega_x - x'\omega_y.$$

The corresponding formulas for components of velocity parallel to the  $x$  and  $y$  axes are, respectively,

$$v_x = \frac{dx}{dt} = \frac{dx_1}{dt} + z'\omega_y - y'\omega_z,$$

$$v_y = \frac{dy}{dt} = \frac{dy_1}{dt} + x'\omega_z - z'\omega_x.$$

When the point  $O_1$  of the body is a fixed point in space, and is taken as origin, the components of the velocity of any point  $P(x, y, z)$  of the body become by this formula

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y.$$

These formulas will be used in a later article.

**171. Kinetic Energy of a Body with One Fixed Point.** It was proved in § 134 that the work done on a body by all impressed forces acting on it in bringing it from rest to any state of motion is equal to  $\frac{1}{2} \int v^2 dM$ , where  $dM$  is the mass of an element of the body and  $v$  is its velocity in the given state of motion. In § 135, this work was defined as the kinetic energy of the body in that state of motion. Since the velocity  $v$  of any element may be resolved into components,  $v_x, v_y, v_z$ , the expression for kinetic energy may be written in the form

$$K.E. = \frac{1}{2} \int (v_x^2 + v_y^2 + v_z^2) dM.$$

In the case of a body with one point fixed, it was shown that the motion is equivalent to a simultaneous rotation about three rectangular axes. If we use these axes as coordinate axes we may express the components of velocity of any element in terms of its coordinates and the angular velocities about the axes in the form shown in § 170,

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y.$$

Hence, in this case, the kinetic energy of the body may be expressed in the form

$$\begin{aligned}
\text{K.E.} &= \frac{1}{2} \int [(z\omega_y - y\omega_z)^2 + (x\omega_z - z\omega_x)^2 + (y\omega_x - x\omega_y)^2] dM \\
&= \frac{1}{2} \int [(z^2 + y^2)\omega_x^2 + (x^2 + z^2)\omega_y^2 + (y^2 + x^2)\omega_z^2] dM \\
&\quad - \int [zy\omega_y\omega_z + xz\omega_z\omega_x + xy\omega_x\omega_y] dM \\
&= \frac{1}{2} [I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2] - \omega_y\omega_z \int yz dM - \omega_x\omega_z \int xz dM \\
&\quad - \omega_x\omega_y \int xy dM.
\end{aligned}$$

If two of the coordinate planes are planes of symmetry of the body, or, in any case, if the chosen axes are the principal axes of the body through the fixed point, the products of inertia all vanish, and the formula reduces to the simpler form

$$\text{K.E.} = \frac{1}{2} [I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2].$$

**172. Kinetic Energy of a Body in Any Motion.** Let us use the method of § 170, and let  $O_1$  denote the centroid of the body. Call the components of the velocity of the centroid,  $\bar{v}_x$ ,  $\bar{v}_y$ ,  $\bar{v}_z$ , and, to avoid primes, let  $(x, y, z)$  instead of  $(x', y', z')$  denote a point of the body referred to the moving axes. Then (§ 170) we shall have

$$v_x = \bar{v}_x + z\omega_y - y\omega_z,$$

and similar formulas for  $v_y$  and  $v_z$ .

In the expression for kinetic energy,

$$\text{K.E.} = \frac{1}{2} \int (v_x^2 + v_y^2 + v_z^2) dM;$$

the first term is

$$\begin{aligned}
&\frac{1}{2} \int v_x^2 dM \\
&= \frac{1}{2} \int (\bar{v}_x^2 + z^2\omega_y^2 + y^2\omega_z^2 + 2z\bar{v}_x\omega_y - 2y\bar{v}_x\omega_z - 2yz\omega_y\omega_z) dM \\
&= \frac{1}{2} \int (\bar{v}_x^2 + z^2\omega_y^2 + y^2\omega_z^2) dM - \omega_y\omega_z \int yz dM.
\end{aligned}$$

The term  $\int z\bar{v}_x\omega_y dM = \bar{v}_x\omega_y \int z dM$  vanishes, since  $z$  is measured from a plane through the centroid. For a similar reason  $\int y\bar{v}_x\omega_z dM$  vanishes. We may then write

$$\frac{1}{2} \int v_x^2 dM = \frac{1}{2} \bar{v}_x^2 M + \frac{1}{2} \omega_y^2 \int z^2 dM + \frac{1}{2} \omega_z^2 \int y^2 dM - \omega_y\omega_z \int yz dM.$$

Writing the corresponding expressions for  $\frac{1}{2} \int v_y^2 dM$  and  $\frac{1}{2} \int v_z^2 dM$ ,

and adding the three expressions, we get for the kinetic energy of the body,

$$\begin{aligned} \text{K.E.} &= \frac{1}{2}M(\bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2) + \frac{1}{2}\omega_x^2 \int (y^2 + z^2) dM \\ &\quad + \frac{1}{2}\omega_y^2 \int (z^2 + x^2) dM + \frac{1}{2}\omega_z^2 \int (x^2 + y^2) dM \\ &\quad - \omega_x \omega_y \int xy dM - \omega_y \omega_z \int yz dM - \omega_x \omega_z \int xz dM \\ &= \frac{1}{2}M\bar{v}^2 + \frac{1}{2}I_x \omega_x^2 + \frac{1}{2}I_y \omega_y^2 + \frac{1}{2}I_z \omega_z^2 - \omega_x \omega_y \int xy dM \\ &\quad - \omega_y \omega_z \int yz dM - \omega_x \omega_z \int xz dM. \end{aligned}$$

Again, if the moving axes at the given instant coincide with the principal axes of the body through the centroid, the products of inertia vanish and

$$\text{K.E.} = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}\{I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2\}.$$

**173. Momentum of a Particle. Impulse.** The product of the mass of a particle and its velocity at a given instant is called the *momentum* of the particle at that instant. Since the velocity is a directed, or vector, quantity, the momentum is also thus regarded, the direction of the momentum being that of the velocity. Thus, if  $m$  is the mass of a particle and  $v$  its velocity, with components  $v_x, v_y, v_z$ , parallel to the co-ordinate axes, the momentum of the particle is  $mv$ , laid off from the particle and directed along the tangent to its path, and the momenta in the directions of the coordinate axes are  $mv_x, mv_y, mv_z$ .

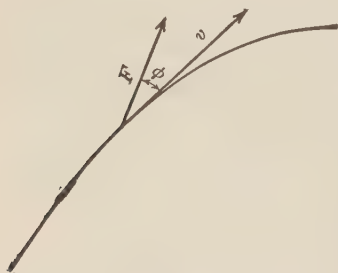


FIG. 364

If the resultant force acting on the particle at any instant is  $F$ , making an angle  $\phi$  with the vector  $v$ , Fig. 364, we shall have  $F \cos \phi = mdv/dt$ , or

$$F \cos \phi \, dt = mdv.$$

Integrating this equation between two positions in which the velocities are  $v_1$  and  $v_2$  and the time changes from  $t_1$  to  $t_2$ , there

results

$$\int_{t_1}^{t_2} F \cos \phi \, dt = m(v_2 - v_1).$$

The integral,  $\int F \cos \phi \, dt$ , is called the *impulse* of the force for the given period of time. If  $F$  is constant in magnitude and is tangent to the curve, making  $\phi = 0$ , the impulse becomes  $F(t_2 - t_1)$ , or the product of the force and the time.

In any case, the impulse of a force acting on a particle for an interval of time is equal to the increase in the scalar value of the momentum of the particle in that interval.

**174. Moment of Momentum of a Particle with Respect to a Line.** In § 173, the momentum of a particle has been defined as a vector laid off from the particle in the direction of its velocity and equal to the product of the mass of the particle and its velocity. The moment of this vector with respect to any line is called the *moment of momentum* of the particle with respect to that line. Thus, in Fig. 365, if  $m$  is the mass of the particle and  $v_h$  is the projection of its velocity upon a plane perpendicular to a given line,  $L$ ; the product,  $pmv_h$ , where  $p$  is the perpendicular distance between  $v_h$  and  $L$ , is, with proper sign, the moment of momentum of the particle with respect to  $L$ . If  $L$  is a directed line, the sign to be used with the preceding product is positive when the direction of rotation of  $v_h$ , viewed from the positive end of the line, appears counterclockwise with respect to the line. Thus the plus sign is used in Fig. 365.

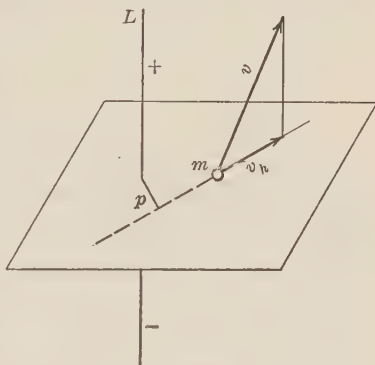


FIG. 365

The term *angular momentum* is often used with the same meaning as moment of momentum.

Referred to rectangular axes, let the coordinates of a particle of mass  $m$  be  $(x, y, z)$ , and let the components of its velocity parallel to the axes be  $v_x, v_y, v_z$  (Fig. 366). Then, since the moment of  $v_h$

with respect to  $OY$  is equal to the sum of the moments of its components, the moment of momentum of the particle with respect to  $OY$  is  $m(zv_x - xv_z)$ . Similarly, the moment of momentum of the particle with respect to  $OX$  may be shown to be  $m(yv_z - zv_y)$ , and that with respect to  $OZ$  may be shown to be  $m(xv_y - yv_x)$ .

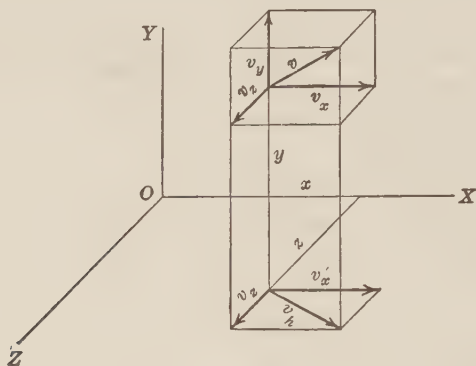


FIG. 366

### 175. Moment of Momentum of a Body with Respect to a Line.

The moment of momentum of a body with respect to a line is defined to be the sum of the moments of momentum of the particles of the body with respect to that line. The symbols  $h_x$ ,  $h_y$ ,  $h_z$  will be used to denote the moments of momentum of a body with respect to the  $x$ ,  $y$ , and  $z$  axes respectively. Then if a particle of the body with coordinates  $x$ ,  $y$ ,  $z$ , has velocity components  $v_x$ ,  $v_y$ ,  $v_z$ , and mass  $dM$ , we have from the definition and from the preceding article,

$$h_x = \int (yv_z - zv_y) dM,$$

$$h_y = \int (zv_x - xv_z) dM,$$

$$h_z = \int (xv_y - yv_x) dM.$$

**176. Moment of Momentum a Vector Quantity.** The moment of momentum of a body with respect to a line may be represented by a vector laid off along the line. To prove this we need only to show that when the moments of momentum of a body with



respect to three rectangular axes are so represented, the moment of momentum of the body with respect to any other line through their intersection may be obtained by adding the components along this line of the vectors along the three axes.

Let vectors equal to  $h_x$ ,  $h_y$ ,  $h_z$  be laid off on the respective axes, and let  $OX'$  and  $OY'$  be axes in the  $xy$  plane making the angle  $\theta$  with  $OX$  and  $OY$  respectively, Fig. 367. Then any

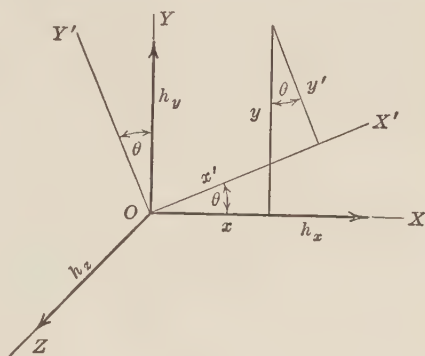


FIG. 367

point  $(x, y, z)$  of the body is given, in the new system of coordinates,  $(x', y', z')$ , by the equations

$$x' = x \cos \theta + y \sin \theta, \quad y' = y \cos \theta - x \sin \theta, \quad z' = z.$$

Moreover, differentiating these formulas, we get

$$v_x' = v_x \cos \theta + v_y \sin \theta, \quad v_y' = v_y \cos \theta - v_x \sin \theta, \quad v_z' = v_z.$$

The component along  $OX'$  of the three vectors  $h_x$ ,  $h_y$ ,  $h_z$  is

$$\begin{aligned} X' &= h_x \cos \theta + h_y \sin \theta \\ &= \int [(y v_z - z v_y) \cos \theta + (z v_x - x v_z) \sin \theta] dM \\ &= \int [(y \cos \theta - x \sin \theta) v_z - z (v_y \cos \theta - v_x \sin \theta)] dM \\ &= \int (y' v_z' - z' v_y') dM, \end{aligned}$$

which is the correct form for  $h_{x'}$ . In the same manner, the vectors used to represent  $h_{x'}$  and  $h_z$  may be combined to obtain a vector which correctly represents the moment of momentum of the body with respect to a line in the plane of  $OX'$  and  $OZ$ . Thus, having given the moments of momentum of a body with respect to three coordinate axes, by representing these moments of momentum as vectors laid off on the respective axes and resolving them into components in the vector method, we may obtain the correct value for the moment of momentum with

respect to any line through the intersection of the three axes. The vector representation of moment of momentum is therefore valid.

### 177. Moment of Momentum of a Body with Respect to a Point.

If the vectors representing the moments of momentum of a body with respect to any three rectangular axes through a point be combined into a single vector, the resultant vector represents the moment of momentum of the body with respect to the line on which this vector falls. Since this vector may be resolved into components along *any* three rectangular axes through the given point, it follows that the combination of the moments of momentum of the body with respect to three rectangular axes through a point gives the same result as the combination of the moments of momentum of the body with respect to any other three rectangular axes through the same point. The resultant vector may therefore be said to represent the *resultant moment of momentum* of the body at the point, or to be the *moment of momentum of the body with respect to the point*.

### 178. Moment of Momentum of a Body with One Point Fixed, in Terms of Angular Velocity.

Let a body with fixed point  $O$  be rotating about the instantaneous axis  $OC$  (Fig. 368), with angular velocity  $\omega$ . Using the vector representation of angular velocity, resolve  $\omega$  into component angular velocities  $\omega_x, \omega_y, \omega_z$  about the axes  $OX, OY, OZ$ , respectively. Let  $dM$  be the mass of an element of the body at  $(x, y, z)$ . The components  $v_x, v_y, v_z$  of the

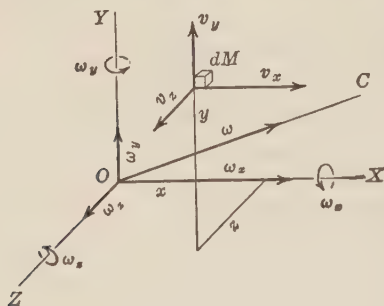


FIG. 368

velocity of  $dM$  may be written (§ 170) in the form

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y.$$

The moment of momentum with respect to the  $x$  axis is then

$$\begin{aligned} h_x &= \int (yv_z - zv_y) dM \\ &= \int [y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x)] dM \\ &= \int [(y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z] dM, \end{aligned}$$

or

$$h_x = I_x\omega_x - \omega_y \int xy dM - \omega_z \int xz dM.$$

Similarly,

$$\begin{aligned} h_y &= I_y\omega_y - \omega_z \int yz dM - \omega_x \int yx dM, \\ h_z &= I_z\omega_z - \omega_x \int zx dM - \omega_y \int zy dM. \end{aligned}$$

**179. Illustration.** Suppose a sphere to be revolving about the  $x$  axis with angular velocity  $\omega$ , Fig. 369. Then, since  $\omega_x = \omega$ ,  $\omega_y = 0$ ,  $\omega_z = 0$ , the expressions for  $v_x$ ,  $v_y$ ,  $v_z$  become

$$\begin{aligned} v_x &= 0, \\ v_y &= -z\omega, \\ v_z &= y\omega. \end{aligned}$$

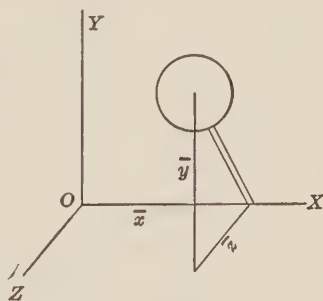


FIG. 369

Then

$$h_x = I_x\omega, \quad h_y = -\omega \int xy dM, \quad h_z = -\omega \int xz dM.$$

It was shown in § 155 that for a sphere  $\int xy dM = \bar{x}\bar{y}M$ , etc., hence

$$h_x = I_x\omega, \quad h_y = -\omega\bar{x}\bar{y}M, \quad h_z = -\omega\bar{x}\bar{z}M.$$

It is interesting here to note that the vector representing the resultant moment of momentum of the sphere at  $O$  does not coincide with the axis of rotation of the body unless this axis either passes through the center of the sphere, in which case both  $\bar{y}$  and  $\bar{z}$  are zero, or else when the center moves in the  $yz$  plane, making  $\bar{x} = 0$ . It may be noted therefore that the vector which represents the moment of momentum of a body with respect to a point does not in general represent the moment of momentum of

the body with respect to another point on the line of the vector. In other words, the vector which represents the moment of momentum of a body with respect to a point cannot be moved along the line of the vector to another point.

The vector representing the moment of momentum of a body with respect to a line may, of course, be moved at will along the line.

### PROBLEMS

468. The sphere of Fig. 370 weighs 50 lbs. and has a radius of 4 in. It is rotating about the  $x$  axis at 120 r.p.m.; and at a given instant its center is on the axis  $OY$ . Find at this instant the moments of momentum of the sphere

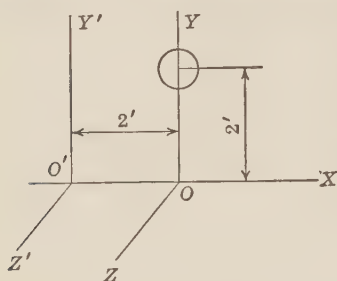


FIG. 370

with respect to  $OX$ ,  $OY$ , and  $OZ$ , and also with respect to  $O'Y'$  and  $O'Z'$ , which are parallel to  $OY$  and  $OZ$ , respectively. From these results find the moments of momentum of the sphere with respect to  $O$  and  $O'$ . Represent them by vectors.

469. Show that if a body has plane motion, the moment of momentum of the body with respect to the instantaneous axis of rotation at any instant is equal to the product of the angular velocity of the body and the moment of

inertia of the body with respect to the instantaneous axis of rotation at the instant.

470. A sphere of mass  $M$  and radius  $r$  has plane motion with angular velocity  $\omega$ . At a given instant its center is distant  $c$  from the instantaneous axis of rotation. Show that at that instant the moment of momentum of the sphere with respect to the instantaneous axis of rotation is  $M\omega(c^2 + 2r^2/5)$ .

**180. Moment of Momentum of a Body in Any Motion.** As in § 172, choose a set of fixed axes,  $OX$ ,  $OY$ ,  $OZ$ , and a set of moving axes parallel to the fixed axes and having the centroid of the body as origin, Fig. 371. We shall then have (§ 172)

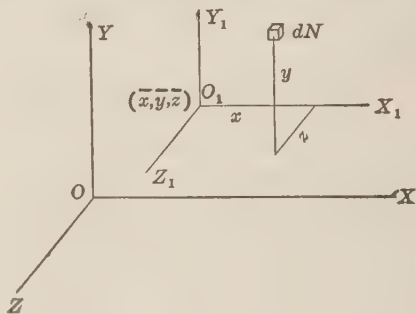


FIG. 371

$$v_y = \bar{v}_y + x\omega_z - z\omega_x,$$

$$v_z = \bar{v}_z + y\omega_x - x\omega_y,$$

and

$$h_x = \int [(y + \bar{y})(\bar{v}_z + y\omega_x - x\omega_y) - (z + \bar{z})(\bar{v}_y + x\omega_z - z\omega_x)] dM.$$

Since  $x$ ,  $y$ , and  $z$  in this expression are measured from planes through the centroid, the terms

$$\int y\bar{v}_z dM, \quad \int \bar{y}\omega_x y dM, \quad \int \bar{y}x\omega_y dM, \\ \int z\bar{v}_y dM, \quad \int \bar{z}x\omega_z dM, \quad \text{and} \quad \int \bar{z}z\omega_x dM$$

all vanish, and the value of  $h_x$  becomes

$$h_x = \omega_x \int (y^2 + z^2) dM - \omega_y \int xy dM + \bar{y}\bar{v}_z M - \omega_z \int xz dM - \bar{z}\bar{v}_y M \\ = M(\bar{y}\bar{v}_z - \bar{z}\bar{v}_y) + I_x \omega_x - \omega_y \int xy dM - \omega_z \int xz dM.$$

The first of these terms is equal to the moment of momentum of a particle of mass  $M$ , located at and moving with the centroid; and the three remaining terms are the value of the moment of momentum of the body with respect to  $O_1X_1$ , with  $O_1$  regarded as at rest. (Compare § 178.) Hence *the moment of momentum of a body in any motion, with respect to any line, is equal to the moment of momentum with respect to the line, of a particle of mass equal to the mass of the body, located at and moving with the centroid, plus the moment of momentum of the body with respect to a parallel line through the centroid, relative to the centroid as at rest.*

### 181. Relation between Torque and Moment of Momentum.

Let  $dM$  be the mass of an element of a body in motion, and  $a_x$ ,  $a_y$ ,  $a_z$  the components of its acceleration parallel to the  $x$ ,  $y$ , and  $z$  axes respectively, Fig. 372. By D'Alembert's Principle, the sum of the moments of the impressed forces about any axis may be equated to the sum of the moments of the effective forces about that axis. The effective force acting on  $dM$ , located at  $(x, y, z)$ , has components  $a_x dM$ ,  $a_y dM$ ,  $a_z dM$  parallel to the

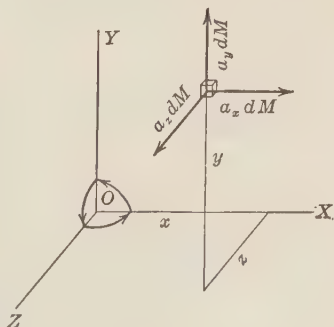


FIG. 372

axes. Hence, calling the torques due to the impressed forces about the axes respectively  $L_x$ ,  $L_y$ ,  $L_z$ , we may write

$$L_x = \int (ya_z - za_y)dM,$$

$$L_y = \int (za_x - xa_z)dM,$$

$$L_z = \int (xa_y - ya_x)dM.$$

Now

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}, \quad a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt};$$

and

$$\frac{d}{dt}(yv_z - zv_y) = ya_z + v_yv_z - za_y - v_zv_y = ya_z - za_y.$$

Therefore

$$L_x = \int \frac{d}{dt}(yv_z - zv_y)dM = \frac{d}{dt} \int (yv_z - zv_y)dM = \frac{dh_x}{dt}.$$

Hence we have

$$L_x = \frac{dh_x}{dt}, \quad L_y = \frac{dh_y}{dt}, \quad L_z = \frac{dh_z}{dt},$$

That is, *the sum of the moments about any axis of the impressed forces acting on any body is equal to the rate of change of the moment of momentum of the body with respect to that axis.*

From this theorem, it follows immediately that if the torque of impressed forces about any axis remains zero throughout the motion of the body, the moment of momentum of the body about that axis remains constant.

#### PROBLEMS

471. Two small weights are placed on a frictionless horizontal rod, which is made to rotate in a horizontal plane, Fig. 373. When the frame has an angular velocity  $\omega_0$ , the distance of each of the particles from the axis of rotation is  $r_0$ . Assuming that there are no external forces to change the speed of rotation, and that the mass of the frame is negligible, show that as the weights recede from the axis, the angular velocity will decrease according to the law  $\omega/\omega_0 = r_0^2/r^2$ , where  $\omega$  is the angular velocity when the distance of each weight from the axis is  $r$ .

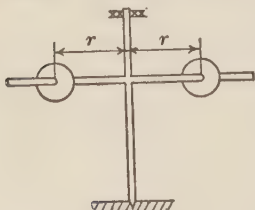


FIG. 373



472. In the preceding problem assume the mass of each particle to be  $M$  and the moment of inertia of the frame with respect to the axis of rotation to be  $I$ , and then show that if there is no impressed torque on the rotating system, the relation between  $r$  and  $\omega$  is given by the equation

$$\frac{\omega}{\omega_0} = \frac{I + 2Mr_0^2}{I + 2Mr^2}.$$

### 182. Rate of Change of a Vector Rotating about a Fixed Point.

Let a vector of variable length and direction be rotating about a fixed point  $O$ , which is the initial point of the vector, Fig. 374. When the vector coincides with a fixed line  $OX$ , let its length be  $h$ .

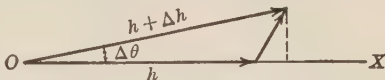


FIG. 374

When it has turned through an angle  $\Delta\theta$  from this position, let its length be  $h + \Delta h$ . Then the vector increase in the direction  $OX$  is

$$(h + \Delta h) \cos \Delta\theta - h, \quad \text{or} \quad \Delta h \cos \Delta\theta - h(1 - \cos \Delta\theta),$$

and the rate of change in this direction is

$$\lim_{\Delta t \rightarrow 0} \left[ \frac{\Delta h}{\Delta t} \cos \Delta\theta - \frac{h 2 \sin\left(\frac{\Delta\theta}{2}\right) \frac{\Delta\theta}{2}}{2 \frac{\Delta\theta}{2}} \frac{\Delta\theta}{\Delta t} \sin\left(\frac{\Delta\theta}{2}\right) \right] = \frac{dh}{dt}.$$

The rate of change of the vector in the direction perpendicular to  $OX$ , is

$$\lim_{\Delta t \rightarrow 0} \frac{(h + \Delta h) \sin \Delta\theta}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(h + \Delta h) \sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta t} = h \frac{d\theta}{dt}.$$

Hence, if a vector  $h$  is rotating about a fixed point with angular velocity  $\omega$ , the rate of change in the direction of the vector is  $dh/dt$ , and the rate of change perpendicular to the vector, in the direction of rotation, is  $h\omega$ .

**183. Rate of Change of Moment of Momentum about Fixed Axes, of a Body with One Fixed Point, in Terms of the Moments of Momentum of the Body about Rotating Axes.** Let a body with fixed point  $O$ , Fig. 375, have moments of momentum  $h_1, h_2, h_3$

about rotating axes  $OX_1$ ,  $OY_1$ ,  $OZ_1$  respectively. Let a set of fixed axes  $OX$ ,  $OY$ ,  $OZ$  be chosen, which at the instant under consideration coincide with the rotating axes. The rotating

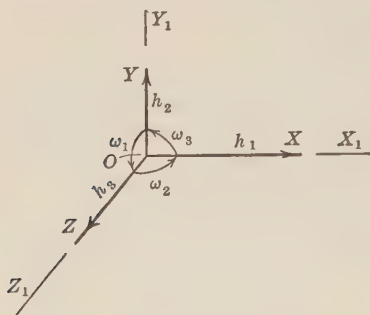


FIG. 375

axes may be regarded as a body moving with one point fixed and, by § 169, this motion is one of rotation about some line through the fixed point  $O$ . Let the angular velocity of this motion be resolved into components  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$ , respectively, about the fixed axes  $OX$ ,  $OY$ ,  $OZ$ . The body whose motion is being studied will, at the

given instant, have the same angular velocities about the rotating axes as about the fixed axes; and therefore the moments of momentum with respect to the fixed and rotating axes will, at the given instant, coincide in value.

The vectors representing the moments of momentum with respect to the rotating axes will have the same angular velocities about the fixed axes as the frame of moving axes, namely,  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$ . Hence, by § 182, Fig. 375, we have

$$\frac{dh_x}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2,$$

$$\frac{dh_y}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3,$$

$$\frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1.$$

**184. The Gyroscopic Couple.** Let Fig. 376 represent a solid of revolution, the axis of the solid being  $OX_1$ , where  $O$  is the centroid of the body. Suppose that the body revolves about  $OX_1$  with constant angular velocity  $\omega$ , and suppose that forces act upon the body that cause the axis  $OX_1$  to rotate about a fixed axis  $OY_1$  perpendicular to  $OX_1$  with a constant angular velocity  $\omega'$ . Choose a third axis  $OZ_1$ , perpendicular to  $OX_1$  and  $OY_1$ . Then

$OX_1$ ,  $OY_1$ , and  $OZ_1$  will be called the *spin axis*, the *precession axis*, and the *torque axis*, respectively.

Let  $I$  and  $I'$  be the moments of inertia of the body with respect to  $OX_1$  and  $OY_1$  (or  $OZ_1$ ), respectively. Choose fixed axes  $OX$ ,  $OY$ ,  $OZ$  which at the given instant coincide with the rotating axes  $OX_1$ ,  $OY_1$ ,  $OZ_1$  respectively. Since  $OY_1$  is to remain at rest, the angular velocities of the frame of the moving axes about the fixed axes are respectively  $\omega_1 = 0$ ,  $\omega_2 = \omega'$ ,  $\omega_3 = 0$ .

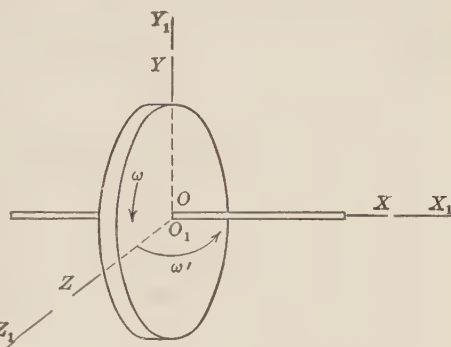


FIG. 376

Since the body is symmetric with respect to the planes  $X_1Y_1$  and  $X_1Z_1$ , all products of inertia referred to the moving axes vanish, and the moments of momentum with respect to the moving axes are (§ 178)

$$h_1 = I\omega, \quad h_2 = I'\omega', \quad h_3 = I'0 = 0.$$

In order to produce the motion described, we must have:

(1) The sum of the components of all of the impressed forces parallel to each of the fixed axes must vanish, since the centroid is at rest (§ 163).

(2) The sum of the moments of all impressed forces about each of the fixed axes must be equal to the rate of change of the moment of momentum of the body about that axis (§ 181).

It follows that the resultant of all impressed forces must be a couple such that the components of its moment about the fixed axes are (§ 183)

$$L_x = \frac{dh_x}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0,$$

$$L_y = \frac{dh_y}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 = 0,$$

$$L_z = \frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 = -I\omega\omega'.$$

Hence, in order to produce the motion described above, it is necessary that at each instant the resultant of the impressed forces be a couple in a plane perpendicular to the torque axis, with a moment equal to  $-I\omega\omega'$ .

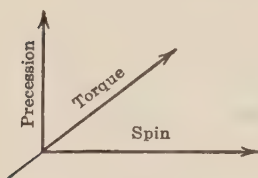


FIG. 377

The couple of moment  $I\omega\omega'$  is called the *gyroscopic couple*.

The directions of rotation and the direction of the torque may be determined from Fig. 377, in which an arrow points along each axis in the direction in which an observer must look in order that the rotation or the torque about that axis should appear clockwise. It is seen from the figure that *the direction of precession is such as to turn the spin vector toward the torque vector*.

### PROBLEMS

473. A disk of radius 6 in. and weight 40 lbs. is centrally mounted on an axis of length 1 ft., of negligible weight, and is set spinning on this axis at 300 r.p.m. The axis is placed with its ends on a smooth, horizontal, circular track and given a motion of 20 r.p.m. about a vertical axis through its center (Fig. 378). Find the pressures of the track on the ends of the axis.

474. With the same rate of spin, what rate of precession would reduce  $P_2$  to 0 in the preceding problem? What happens if the precession rate exceeds this value?

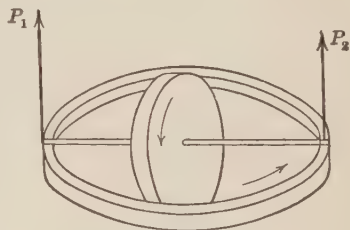


FIG. 378

**185. The Gyroscope; Simplest Case.** The simple gyroscope is illustrated in Fig. 379. It consists of a solid of revolution mounted upon an axis in line with its geometric axis, which is free to turn in bearings in a frame which is supported at one point in the geometric axis.

Let the weight of the body be  $W$ , its mass  $M$ , and let the distance from the centroid to the point of support,  $O$ , be  $l$ . The weight of the ring in which the wheel is mounted is supposed to be negligible in comparison with the weight of the wheel.

Suppose that the wheel is given an angular velocity  $\omega$  about its

geometric axis (spin axis), and that at the same time the spin axis is set in rotation about a vertical axis (precession axis) through the point of support. The simplest case is that in which both angular velocities are kept constant and the spin axis is kept horizontal. We shall assume that the impressed forces are applied to make this motion possible, and we shall try to determine the required system of forces.

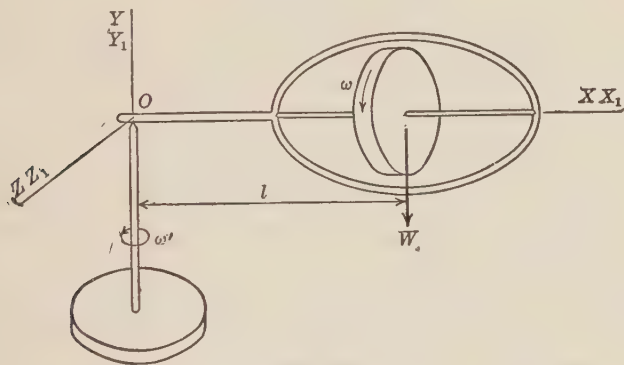


FIG. 379

Assume rotating axes  $OX_1$ ,  $OY_1$ ,  $OZ_1$  to coincide with the spin axis, the precession axis, and a third axis, perpendicular to these two, respectively. At a given instant, assume fixed axes,  $OX$ ,  $OY$ ,  $OZ$  which coincide with the moving axes at that instant. Let  $I$  be the moment of inertia of the body about the spin axis, and  $I'$  the moment of inertia about the precession axis. Let  $\omega$  be the angular velocity of spin and  $\omega'$  the angular velocity of the spin axis about the precession axis. If  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  are the angular velocities of the frame of rotating axes with respect to the fixed axes, then

$$\omega_1 = 0, \quad \omega_2 = \omega', \quad \omega_3 = 0.$$

If  $h_1$ ,  $h_2$ ,  $h_3$  are the moments of momentum of the body about the rotating axes, then

$$h_1 = I\omega, \quad h_2 = I'\omega', \quad h_3 = 0.$$

Letting  $X$ ,  $Y$ ,  $Z$  and  $L_x$ ,  $L_y$ ,  $L_z$ , respectively, be the resultant components of the impressed forces parallel to the axes and the

torques of the impressed forces about these axes, we may apply the theorems of §§ 163, 181, and 183, and obtain

$$X = -LM\omega'^2, \quad Y = 0, \quad Z = 0,$$

$$L_x = \frac{dh_x}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0,$$

$$L_y = \frac{dh_y}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 = 0,$$

$$L_z = \frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 = -I\omega\omega'.$$

From these equations it is clear that the assumed motion is possible under no external forces except the weight of the body and the reaction of the support, provided only that  $\omega$  and  $\omega'$  have such values that  $-Wl = -I\omega\omega'$  and that the base is rigidly fixed to withstand the force  $LM\omega'^2$ .

Assuming that there is no friction, the motion will therefore continue with constant spin velocity  $\omega$  and constant precessional velocity  $\omega'$ , provided  $I\omega\omega' = Wl$  at the start.

The gyroscope in this case is said to have *steady precession*.

### PROBLEMS

475. A gyroscope consists of a wheel whose radius of gyration about its axis is 3 in., and the point of support is 4 in. from the centroid of the wheel. If it is set rotating at 300 r.p.m., with axis horizontal, what rate of precession should it be given in order that the axis may remain in the horizontal plane?

**186. Tendency of a Rotating Body to Maintain Its Plane of Motion.** Consider again a body in the form of a solid of revolution, rotating about its geometric axis, the impressed forces being such that the axis of rotation (spin axis) remains always in the same plane.

Suppose the centroid,  $O_1$ , to be in motion in this plane. Choose a set of moving axes,  $O_1X_1$ ,  $O_1Y_1$ ,  $O_1Z_1$  so that  $O_1X_1$  is the spin axis of the body and  $O_1Y_1$  is perpendicular to the plane in which the spin axis moves, Fig. 380. Choose a set of fixed axes  $OX$ ,  $OY$ ,  $OZ$ , with which, at the given instant, the moving axes coincide. The frame of the moving axes then has a motion made up of the motion of the centroid and a motion of rotation



about the fixed axes (§ 170). Since in the motion under discussion the axes  $OY$  and  $O_1Y_1$  remain always perpendicular to the same plane, and hence parallel to each other, the only motion of rotation of the frame of the moving axes relative to the fixed axes is one of rotation about  $OY$ . If then the angular velocity of rotation of the spin axis in its plane is  $\omega'$ , and if  $\omega_1, \omega_2, \omega_3$  are the

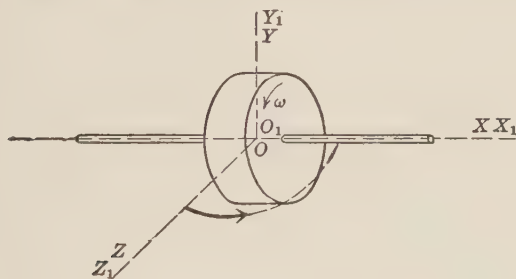


FIG. 380

components of the angular velocity of the moving axes with respect to the fixed axes, we have

$$\omega_1 = 0, \quad \omega_2 = \omega', \quad \omega_3 = 0.$$

If  $\omega$  denotes the angular velocity of spin of the body,  $I$  the moment of inertia of the body about the spin axis,  $I'$  the moment of inertia about the axis  $O_1Y_1$ , and  $h_1, h_2, h_3$  the moments of momentum of the body with respect to  $O_1X_1, O_1Y_1, O_1Z_1$  respectively, then (§ 184) we shall have

$$h_1 = I\omega, \quad h_2 = I'\omega', \quad h_3 = 0.$$

In § 180 we derived the formula

$$h_x = M(\bar{y}\bar{v}_z - \bar{z}\bar{v}_y) + I_x\omega_x - \omega_y \int xy dM - \omega_z \int xz dM,$$

in which the last three terms represent the moment of momentum of the body with respect to a line through the centroid parallel to the  $x$  axis, relative to the centroid as at rest. Let this axis be represented by  $O_1X'$ , and the corresponding  $h$  by  $h_x'$ . We then have

$$h_x = M(\bar{y}\bar{v}_z - \bar{x}\bar{v}_y) + h_x'.$$

Taking the derivatives of both sides of this equation with respect to  $t$ , we obtain

$$\frac{dh_x}{dt} = M(\bar{y}\bar{a}_x - \bar{z}\bar{a}_y) + \frac{dh_x'}{dt}.$$

In the case under discussion,  $O_1$  coincides with  $O$  at the given instant; hence both  $\bar{y}$  and  $\bar{z}$  are zero, and

$$\frac{dh_x}{dt} = \frac{dh_x'}{dt}.$$

This means that the rate of change of the moment of momentum of the body with respect to a line through the centroid of the body is the same as if the centroid were at rest and the body had the same angular motion about the axes. Then using the formulas of § 183 we may write

$$(1) \quad L_x = \frac{dh_x}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = I \frac{d\omega}{dt},$$

$$(2) \quad L_y = \frac{dh_y}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 = I' \frac{d\omega'}{dt},$$

$$(3) \quad L_z = \frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 = -I\omega\omega'.$$

In addition to these three torque equations, we must have for the motion of the centroid (§ 163) the three equations

$$X = Ma_x, \quad Y = Ma_y, \quad Z = Ma_z,$$

where  $X$ ,  $Y$ ,  $Z$  are the resultant components of the impressed forces acting on the body, and  $a_x$ ,  $a_y$ ,  $a_z$  are the components of the acceleration of the centroid of the body.

Of the above three torque equations, (1) is independent of  $\omega'$  and (2) is independent of  $\omega$ , which means that in order to cause a certain angular acceleration of the body with respect to the spin axis, precisely the same torque of impressed forces about the spin axis is required as though the spin axis were at rest; while to accelerate the precession velocity, the same torque about the precession axis is required as though the body had no spin velocity. Equation (3), however, expresses an additional con-

dition that must be fulfilled when the prescribed motion takes place, namely, that there must be a torque of impressed forces about an axis perpendicular to the spin axis and the precession axis that is equal to minus the product of the moment of inertia of the body about the spin axis, the angular velocity of spin, and the angular velocity of precession of the spin axis.

In the above discussion it has been assumed that the body dealt with is a solid of revolution about the spin axis.

It is the requirement of a torque,  $-I\omega\omega'$ , which makes the difference between turning a rotating body and a non-rotating one. For if the body is not rotating about its geometric axis,  $\omega = 0$ , and the torque  $-I\omega\omega'$  vanishes.

If both the spin velocity and the precession velocity are constant, both  $L_x$  and  $L_y$  are zero and the only torque required for this motion is  $L_z = -I\omega\omega'$ . The greater the spin velocity, the smaller is the velocity of precession for a given impressed torque. If there were no spin velocity, no torque would be required to maintain a constant angular velocity  $\omega'$ .

We thus see that a body of revolution when spinning about its geometric axis offers a resistance to any change of its plane of rotation, and that the torque representing this resistance varies directly as the product of the spin velocity and the angular velocity of the plane of motion. This property renders a rapidly rotating wheel, or *gyroscope*, a stabilizing factor which has been given practical application in several engineering inventions, such as stabilizing ships and airplanes, steering torpedoes, balancing cars to run on one rail, the gyro-compass, etc. In these appliances rather complicated devices for restoring equilibrium are necessary, and only a few of the applications that have been attempted have, as yet, become commercially successful. A brief description of two of these applications will be given in later paragraphs.

**187. Upsetting Torque Due to Rotation.** Let a pair of wheels and their axle have a weight  $W$  and a moment of inertia  $I$  about the axis. In Fig. 381 the wheels are represented as rolling along a track which turns to the right.

Let  $F_1$  and  $F_2$  be the horizontal forces and  $P_1$  and  $P_2$  the vertical forces exerted upon the wheels by the track. Since the precession, as viewed from the top, is clockwise, and since the precession must turn the spin axis toward the torque axis (§ 184),

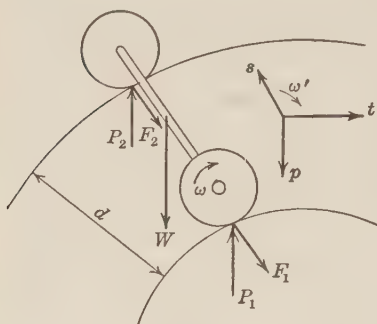


FIG. 381

the spin axis  $s$ , the torque axis  $t$ , and the precession axis  $p$  must be situated as indicated in the figure. Let us assume that the wheels are turning with angular velocity  $\omega$  about their axis and that the axis is turning with angular velocity  $\omega'$  in the horizontal plane. Let the radius of the curve in which the centroid is moving be  $R$ , let  $h$  be the height of the centroid

above the track, and let  $d$  be the distance between the wheels. We may then write the following equations for the motion of the body (§§ 163, 185),

$$(1) \quad P_1 + P_2 = W,$$

$$(2) \quad F_1 + F_2 = \frac{W}{g} R \omega'^2,$$

$$(3) \quad (P_2 - P_1) \frac{d}{2} - (F_1 + F_2)h = I \omega \omega'.$$

From (2) it follows that  $F_1 + F_2$  is positive and then from (3) it follows that  $P_2 > P_1$ . This latter relation would be true even if there were no rotation of the wheels about the axis, for then

$$(P_2 - P_1) \frac{d}{2} = (F_1 + F_2)h.$$

However, equation (3) shows that the difference between  $P_2$  and  $P_1$  is increased by the rotation of the wheels. Hence the rotation of the wheels tend to lift the inner wheel from the track when the wheels are running on a curved track. Thus the upsetting tendency of a car when making a turn is increased by the rotation of the wheels.

## PROBLEMS

476. A wheel is rolling in a straight line with its centroid in the vertical plane containing its path when a slight sideways pressure is exerted near its top which displaces the center from the vertical plane in which it has been moving, Fig. 382. Show that after the pressure is removed, the torque set up by the weight and the reaction of the track causes the wheel to turn from its straight line path toward the side to which the centroid was moved.



FIG. 382

477. Suppose that the pair of wheels and axle of § 187 weigh 1200 lbs., that their radius of gyration is 10 in., and that the radius of the wheels is 15 in. If

the distance between the wheels is 4 ft. and the radius of the curve in which the centroid is moving is 100 ft., at what speed of the centroid would the inner wheel lift from the track? What speed would be required if the wheels were not rotating?

478. Suppose that the wheels of § 187 carry an additional weight  $W'$  which does not rotate, and write the equations for the motion of the system. Show that in this case that the upsetting effect due to the rotation of the wheels when making a turn is relatively less important.

479. A flywheel in the form of a uniform disk with radius 2 ft., thickness 3 in., and weight 1500 lbs. is mounted on a horizontal axle with 2 ft. between bearings. The wheel is making 1200 r.p.m. and the platform on which the wheel is mounted is being rotated in a horizontal plane at 20 r.p.m. Find the vertical reaction at each bearing.

**188. The Steering of a Torpedo.** When a torpedo is launched in the direction of the object of attack, the axis of the rapidly spinning gyroscope which the torpedo carries is set in the direction of the desired motion of the torpedo, a direction which the axis tends strongly to maintain. The gyroscope is so mounted that the torpedo may change its direction relative to the axis of the gyroscope, but as it does so, valves connected with a compressed air chamber are opened, or closed, and by their connection with a steering device, operate to restore the torpedo to its original direction. The path of the torpedo is thus a narrow zigzag path in the original direction in which the torpedo is launched. It is also possible to arrange the steering device so that the course of the torpedo may be changed after traveling a certain distance.

**189. The Gyro-Compass.** The essential features of a gyro-compass consist of a disk, or gyro-wheel, kept in rapid rotation by a motor, and suspended in a mercury bath as indicated in Fig.







inclination of the axis to the earth's surface increases and hence the torque tending to return the spin axis to the westward is increased. The spin axis would then continue the precession beyond the meridian. In order to prevent this, another torque is brought into action which tends to turn the spin axis downward toward the tangent to the meridian. This is accomplished by

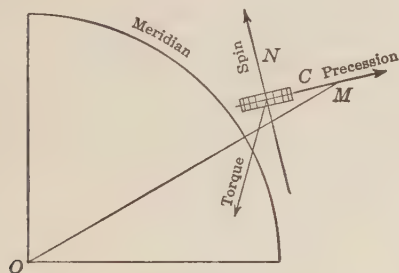


FIG. 385

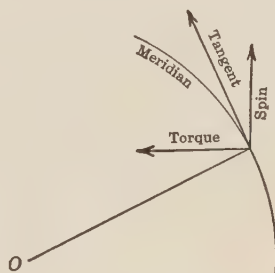


FIG. 386

two jets of air applied in such a way that the torque axis due to them points downward along the vertical, Fig. 386. The jets are regulated by valves which are controlled by a pendulum which hangs vertically and which, by its relation to the gyro frame, determines the relative magnitude of the jets. The two torques, due to the rotation of the earth and to the action of the jets, therefore operate to keep the spin axis pointing to the north.

This explanation attempts to give only the fundamental principles under which the gyro-compass operates. A rather full discussion may be found in H. Crabtree's *Spinning Tops and Gyroscopic Motion* (Longmans), and in R. F. Deimel, *Mechanics of the Gyroscope* (The Macmillan Company).

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480. Suppose that a gyro-compass is mounted on a ship and placed with its spin axis pointing north. Show that if the ship moves north, a torque tending to turn the spin axis to the west will be set up, independent of the torque introduced by the rotation of the earth.

190. **The Gyroscope, Inclined Axis.** In Fig. 387 let the axis of the gyroscope make an angle  $\theta$  with the vertical axis  $OY$  at any instant, where  $O$  is the point of support of the gyroscope. Choose a fixed horizontal axis  $OX$  which lies in the vertical plane con-

taining the axis of the body at the given instant. Let  $OZ$  be perpendicular to  $OX$  and  $OY$ . Let  $OA$  be a fixed axis coinciding with the axis of the body at the given instant, and  $OB$  a fixed axis perpendicular to  $OA$  and  $OZ$ . Let  $OA_1$ ,  $OB_1$ ,  $OZ_1$ , be moving axes,  $OA_1$  coinciding with the geometric axis of the body,  $OB_1$  perpendicular to  $OA_1$  and in a vertical plane, and  $OZ_1$  perpendicular to  $OA_1$  and  $OB_1$ . At the instant selected  $OA_1$ ,  $OB_1$ , and  $OZ_1$  coincide with  $OA$ ,  $OB$ , and  $OZ$  respectively. Let the angular velocity of the body with respect to its own axis (spin velocity) be  $\omega$ , and the angular velocity, with respect to  $OY$ , of the vertical plane containing the axis  $OA$  be  $\omega'$ , and the angular velocity of  $OA_1$  with respect to  $OY$  be  $-d\theta/dt = -\omega''$ .

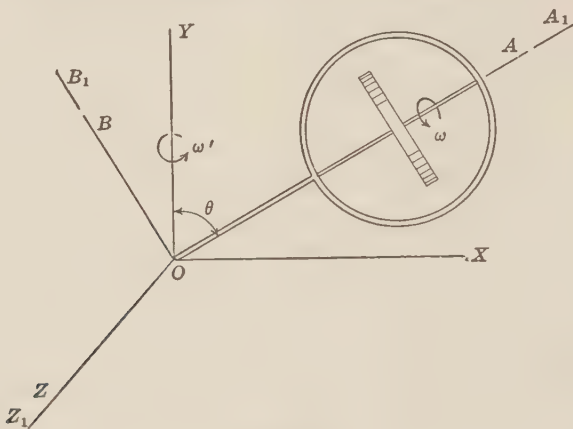


FIG. 387

Resolving  $\omega'$ , which may be represented by a vector along  $OY$ , into components along  $OA$  and  $OB$ , we have  $\omega' \cos \theta$  along  $OA$  and  $\omega' \sin \theta$  along  $OB$ . We then have for the component angular velocities of the body with respect to  $OA$ ,  $OB$ , and  $OZ$ , respectively, the values

$$\omega + \omega' \cos \theta, \quad \omega' \sin \theta, \quad -\omega''.$$

Moreover, the components of the angular velocity of the frame of moving axes,  $OA_1$ ,  $OB_1$ ,  $OZ_1$ , with respect to the fixed axes,  $OA$ ,  $OB$ ,  $OZ$ , respectively are

$$\omega_1 = \omega' \cos \theta, \quad \omega_2 = \omega' \sin \theta, \quad \omega_3 = -\omega''.$$

Let  $I$  and  $I'$  be the moments of inertia of the body with respect to  $OA_1$  and  $OB_1$  (or  $OZ_1$ ), respectively. Then the moments of momentum of the body with respect to  $OA_1$ ,  $OB_1$ , and  $OZ_1$  are, respectively,

$$h_1 = I(\omega + \omega' \cos \theta), \quad h_2 = I'\omega' \sin \theta, \quad h_3 = -I'\omega''.$$

The discussion will be divided into two parts, (a) *steady precession*, where  $\omega$ ,  $\omega'$  and  $\theta$  are all constant, (b) *unsteady precession*, where  $\omega$ ,  $\omega'$ , and  $\theta$  are variable.

(a) **STEADY PRECESSION.** Assume that the impressed forces are such that  $\omega$ ,  $\omega'$ , and  $\theta$  are constant, and hence that  $\omega'' = 0$ .

If  $L_a$ ,  $L_b$ ,  $L_z$  represent the resultant torques of the impressed forces with respect to the axes  $OA$ ,  $OB$ ,  $OZ$ , we shall have

$$L_a = \frac{dh_a}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0,$$

$$L_b = \frac{dh_b}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 = 0,$$

$$\begin{aligned} L_z &= \frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 \\ &= -I(\omega + \omega' \cos \theta)\omega' \sin \theta + I'\omega'^2 \sin \theta \cos \theta. \end{aligned}$$

Moreover, since the centroid moves in a circle of radius  $l \sin \theta$ , where  $l$  is the distance of the centroid from the point of support, the sums of the components of the impressed forces parallel to the axes  $OX$ ,  $OY$ ,  $OZ$  are

$$X = -Ml \sin \theta \omega'^2, \quad Y = 0, \quad Z = 0.$$

It is possible for the assumed motion to occur when the only impressed forces are the weight of the body and the reaction of the support. For, if in this case the reaction of the support has components  $R_x$ ,  $R_y$ ,  $R_z$ , the following equations must be satisfied:

$$R_x = -Ml \sin \theta \omega'^2, \quad R_y = W, \quad R_z = 0.$$

$$L_a = 0, \quad L_b = 0,$$

$$-Wl \sin \theta = -I\omega\omega' \sin \theta - (I - I')\omega'^2 \cos \theta \sin \theta.$$

If the support is rigid, the first two equations will be satisfied, since the support will exert any force required. The last equation

will be satisfied provided

$$(I' - I)\omega'^2 \cos \theta = I\omega\omega' - Wl.$$

Any two of the quantities,  $\omega$ ,  $\omega'$ ,  $\theta$ , may then be assigned at will, and the third determined from the equation, subject to the condition that  $\cos \theta$  must lie between  $-1$  and  $1$ , and that  $\omega'$  must be real. For example, having given  $\theta$  and  $\omega$ , the value of  $\omega'$  is

$$\omega' = \frac{I\omega \pm \{I^2\omega^2 - 4Wl(I' - I) \cos \theta\}^{1/2}}{2(I' - I) \cos \theta}.$$

From this it follows that for given values of  $\omega$  and  $\theta$  there will be two values, one value, or no value of  $\omega'$  according as

$$I^2\omega^2 \gtrless 4Wl(I' - I) \cos \theta.$$

(b) UNSTEADY PRECESSION. When  $\omega$ ,  $\omega'$ , and  $\theta$  are variable, the torque equations become

$$\begin{aligned} L_a &= \frac{dh_a}{dt} = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 \\ &= I \frac{d}{dt}(\omega + \omega' \cos \theta) + I'\omega'\omega'' \sin \theta - I'\omega'\omega'' \sin \theta \\ &= I \frac{d}{dt}(\omega + \omega' \cos \theta), \\ L_b &= \frac{dh_b}{dt} = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 \\ &= I' \left( \frac{d\omega'}{dt} \sin \theta + \omega'\omega'' \cos \theta \right) + I'\omega'\omega'' \cos \theta \\ &\quad - I(\omega + \omega' \cos \theta)\omega'', \\ L_z &= \frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 \\ &= -I' \frac{d\omega''}{dt} - I(\omega + \omega' \cos \theta)\omega' \sin \theta + I'\omega'^2 \sin \theta \cos \theta. \end{aligned}$$

Again let us suppose that the only impressed forces are the weight and the reaction of the support; and let us seek to determine the conditions under which the motion will continue, and

to analyze the motion. We need to consider only the torque equations, for if the support is rigidly fixed, it will exert any needed forces in the directions of the axes. In this case the three torque equations become

$$L_a = I \frac{d}{dt} (\omega + \omega' \cos \theta) = 0,$$

$$L_b = I' \left( \frac{d\omega'}{dt} \sin \theta + \omega' \cos \theta \frac{d\theta}{dt} \right) + I' \omega' \cos \theta \frac{d\theta}{dt} - I(\omega + \omega' \cos \theta) \frac{d\theta}{dt} = 0,$$

$$L_z = -I \frac{d^2\theta}{dt^2} - I(\omega + \omega' \cos \theta) \omega' \sin \theta + I' \omega'^2 \sin \theta \cos \theta = -Wl \sin \theta.$$

The first of these equations shows that  $\omega + \omega' \cos \theta = k$ , a constant. Multiplying the second equation by  $\sin \theta$ , we may write

$$I' \left( \frac{d\omega'}{dt} \sin^2 \theta + 2\omega' \cos \theta \sin \theta \frac{d\theta}{dt} \right) - Ik \sin \theta \frac{d\theta}{dt} = 0,$$

or, since the quantity in the parenthesis is  $d(\omega' \sin^2 \theta)/dt$ , we have, after integration,

$$I' \omega' \sin^2 \theta + Ik \cos \theta = k_1.$$

The three torque equations are therefore equivalent to the equations

$$\omega + \omega' \cos \theta = k,$$

$$I' \omega' \sin^2 \theta + Ik \cos \theta = k_1,$$

$$I \frac{d^2\theta}{dt^2} = \sin \theta [Wl + (I' \omega' \cos \theta - Ik) \omega'].$$

To simplify the problem, let the body be started with

$$\theta = \theta_0 < 90^\circ, \quad \omega = \omega_0 > 0, \quad \omega' = 0, \quad d\theta/dt = 0.$$

Then  $k = \omega_0$  and  $k_1 = I\omega_0 \cos \theta_0$ . The work equation also gives a useful relation, though not an additional independent equation. This equation comes from equating the work done on the body to

its gain in kinetic energy between the start and any other instant. Assuming that there is no friction, the only work done on the body is that done by gravity, which, from  $\theta = \theta_0$  to  $\theta = \theta$ , is  $Wl(\cos \theta_0 - \cos \theta)$ . The gain in kinetic energy (§ 171) for the same change of position is

$$\frac{1}{2}I[(\omega + \omega' \cos \theta)^2 - \omega_0^2] + \frac{1}{2}I'[(\omega' \sin \theta)^2 - 0] \\ + \frac{1}{2}I' \left[ \left( -\frac{d\theta}{dt} \right)^2 - 0 \right],$$

or, since  $\omega + \omega' \cos \theta = \omega_0$ , the gain in kinetic energy is

$$\frac{1}{2}I' \left[ \omega'^2 \sin^2 \theta + \left( \frac{d\theta}{dt} \right)^2 \right].$$

Therefore the work equation becomes

$$Wl(\cos \theta_0 - \cos \theta) = \frac{1}{2}I' \left[ \omega'^2 \sin^2 \theta + \left( \frac{d\theta}{dt} \right)^2 \right].$$

Collecting the preceding four equations, we have

$$(1) \quad \omega + \omega' \cos \theta = \omega_0,$$

$$(2) \quad I'\omega' \sin^2 \theta + I\omega_0(\cos \theta - \cos \theta_0) = 0,$$

$$(3) \quad I \frac{d^2\theta}{dt^2} = \sin \theta [Wl + (I'\omega' \cos \theta - I\omega_0)\omega'],$$

$$(4) \quad \left( \frac{d\theta}{dt} \right)^2 = \omega' \left[ \frac{2Wl}{I\omega_0} - \omega' \right] \sin^2 \theta.$$

Equation (4) is the work equation combined with equation (2). It must be noted that (2) and (4) do not hold for  $\theta = 180^\circ$ , for (2) was derived by multiplying another equation through by  $\sin \theta$ , which would not be valid if  $\sin \theta = 0$ .

Consider the motion just as the body is released. Since at that instant  $\omega' = 0$  and  $\sin \theta$  is positive, the value of  $d^2\theta/dt^2$  is positive. Therefore  $d\theta/dt$  begins by increasing from 0. This means that  $\theta$  begins to increase, or the spin axis begins to fall. Since  $d\theta/dt$  begins to increase, equation (4) shows that  $\omega'$  cannot remain equal to zero. Moreover, it must become positive, for if



it were negative, the right side of (4) would be negative, which is impossible, since the left side is positive. Hence precession begins, and in a positive direction.

Solving (2) for  $\omega'$  and taking the derivative with respect to  $\theta$ , we find

$$\frac{d\omega'}{d\theta} = \frac{I\omega_0}{I' \sin^3 \theta} [\sin^2 \theta_0 + (\cos \theta_0 - \cos \theta)^2].$$

Since  $\omega_0$  and  $\sin^3 \theta$  are positive,  $d\omega'/d\theta$  is positive. Therefore  $\omega'$  must increase as  $\theta$  increases and decrease as  $\theta$  decreases. Hence the rate of precession will increase as the spin axis falls. From (4) it follows that if  $\omega'$  increases to  $2Wl/I\omega_0$  before  $\theta$  increases to  $180^\circ$ ,  $d\theta/dt$  becomes 0, and the spin axis ceases to fall. The substitution of  $2Wl/I\omega_0$  in (2) will determine the value of  $\cos \theta$ . In order that the spin axis stop falling before  $\theta = 180^\circ$ , this value of  $\cos \theta$  must lie between the values  $\cos \theta_0$  and  $-1$ .

Assume that the value of  $\theta$  so determined is less than  $180^\circ$ . Since at the beginning of the motion  $d\theta/dt$  was zero and began to increase, it must have been decreasing when it became zero again and must therefore have become negative at that value of  $\theta$ . Hence  $\theta$  must begin to decrease and the spin axis rises. Since, as has been shown,  $\omega'$  and  $\theta$  increase or decrease together, the value of  $\omega'$  will decrease. This decrease will continue as long as  $\theta$  decreases, that is, until  $d\theta/dt = 0$ , or  $\omega' = 0$ , and hence  $\theta = \theta_0$  [see equation (2)]. The original conditions then exist, except that the spin axis is in a different vertical plane. The spin axis will therefore alternately fall and rise to the initial elevation as the vertical plane containing the spin axis "precesses" about the vertical line through the point of support. As the spin axis falls, the rate of precession increases, and as the spin axis rises, the rate of precession decreases.

**191. Illustration.** Suppose that the gyroscope is composed of a steel disk 1 in. thick, radius 2 in., with center 3 in. from the point of support, and a supporting frame whose weight may be neglected. If the initial conditions are  $\theta_0 = 60^\circ$ ,  $\omega_0 = 1200$  r.p.m.  $= 40\pi$  rad./sec.,  $\omega'_0 = 0$ , what is the subsequent motion?

We have

$$W = 490\pi \left(\frac{1}{36}\right) \left(\frac{1}{12}\right) = 3.563 \text{ lbs.,}$$

$$I = \frac{1}{2} \frac{W}{g} r^2 = \frac{3.563}{64.4} \left(\frac{1}{36}\right) = 0.001537,$$

$$I' = \frac{W}{g} \left( \frac{r^2}{4} + \frac{h^2}{12} + d^2 \right) = \frac{3.563}{32.2} \left( \frac{1}{144} + \frac{1}{1728} + \frac{1}{16} \right) \\ = 0.007749.$$

Setting

$$\omega' = \frac{2Wl}{I\omega_0} = \frac{2(3.563)}{0.001537(40\pi)(4)} = 9.225,$$

and substituting in (2), we may write this equation in the form  
 $0.007749(9.225)(1 - \cos^2 \theta) + 0.001537(40\pi)(\cos \theta - 0.5) = 0,$   
 or

$$\cos^2 \theta - 2.7018 \cos \theta + 0.3509 = 0.$$

Therefore

$$\cos \theta = 0.1363, \quad \theta = 82^\circ 10'.$$

Hence the spin axis alternately falls and rises, oscillating between the values  $\theta = 60^\circ$  and  $\theta = 82^\circ 10'$ , the angular velocity of precession varying at the same time between 0 and 9.225 rad./sec.

### PROBLEMS

481. Show that if the value of  $\omega_0$  is only 300 r.p.m. in the above illustration, with the other initial conditions unchanged, the value of  $\omega'$  will vary between 0 and 36.90 rad./sec., while  $\theta$  varies between  $60^\circ$  and  $151^\circ 10'$ .

482. If the gyroscope is a steel disk 1 in. thick, 3 in. radius, with center 2 in from the point of support, and if  $\theta_0 = 30^\circ$ ,  $\omega_0 = 1200$  r.p.m.,  $\omega'_0 = 0$ , show that  $\theta$  varies from  $30^\circ$  to about  $30^\circ 55'$  while  $\omega'$  varies from 0 to 2.733 rad./sec.

483. If in the preceding problem  $\omega_0 = 120$  r.p.m., with other initial conditions unchanged, show that  $\omega'$  changes from 0 to 27.33 rad./sec., while  $\theta$  varies from  $30^\circ$  to about  $134^\circ 20'$ .

484. By equation (4) of § 190 show that the value of  $\omega'$  which makes  $d\theta/dt = 0$  is very small when  $\omega_0$  is very large; and then show by equation (2), that  $\theta$  varies very little from  $\theta_0$ .

## ANSWERS

1.  $OA = 742$  lbs.,  $OB = 1137$  lbs.      2.  $W = 209$  lbs.,  $T = 174$  lbs.
3.  $OA = 359$  lbs.,  $OB = 416$  lbs.      4. (a)  $P = 518$  lbs.,  $N = 732$  lbs.,  
(b) along the plane  $P = 500$  lbs.,  $N = 866$  lbs.
5.  $W = 5284$  lbs., Tens. in  $AB = 5416$  lbs.      6.  $T = 28.7$  lbs.
7.  $BC = 1.25W$ ,  $AC = 0.75W$ .      8.  $W, 1.732W$ .      9.  $3.54$  lbs.      10.  $80$  lbs.
11.  $28.9$  lbs.,  $-13^\circ 53'$  with  $OX$ .      12.  $77.0$  lbs.,  $266^\circ 53'$  with  $OX$ .
13.  $414$  lbs.,  $E 4^\circ 4' N$ .      14.  $V = 241$  lbs.,  $H = 49.6$  lbs.
15.  $P_2 = 51.8$  lbs.,  $P_3 = 89.6$  lbs.,  $W = 201$  lbs.
16.  $P = 89.7$  lbs.,  $N = 163$  lbs.      17.  $P = 15.1$  lbs.
18.  $P = 90.4/(\cos \theta + 0.2 \sin \theta)$ ;  $11^\circ 19'$ ,  $P = 88.7$  lbs.,  $N = 176$  lbs.
19.  $P = W \sin(\alpha + \theta)/\cos(\varphi - \theta)$ , where  $\theta = \tan^{-1} \mu$ .
20.  $P = W \sin(\alpha - \theta)/\cos(\varphi + \theta)$ , where  $\theta = \tan^{-1} \mu$ .
23. In  $BC$   $1796$  lbs.,  $R_a = 1002$  lbs., in  $DF$   $3771$  lbs.,  $R_c = 2749$  lbs.
24. At  $A$   $3132$  lbs., at  $B$   $2100$  lbs., in  $DF$   $2828$  lbs.,  $R_c = 5099$  lbs.
25. (a)  $234$  lbs., (b)  $222$  lbs., (c)  $226$  lbs.
26.  $299$  lbs., min.  $P = 166$  lbs., at  $56^\circ 15'$  with horizontal.
27. At  $O$   $R = 151.6$  lbs.,  $144.2$  lbs. in  $L$  and  $L'$ ;  $151.6$  lbs. at  $53^\circ 18'$  with  $OX$ ,  $2.854$  in. from  $O$ .
28. In  $ED$   $3.33$  tons,  $R_a = 5.95$  tons, in  $BG$   $6.25$  tons,  $R_c = 4.25$  tons.
29. In  $BC$   $4.95$  tons,  $X_a = 2.48$  tons,  $Y_a = 8.29$  tons, in  $CE$   $4$  tons tension, in  $DE$   $6.93$  tons compression.
32.  $P = 945$  lbs.,  $R_a = 1393$  lbs.
33.  $P = 251$  lbs.,  $X_a = 564$  lbs.,  $Y_a = 286$  lbs.
34. In  $CD$   $100$  lbs.,  $X_a = 64.3$  lbs.,  $Y_a = 430$  lbs.
35.  $CD$   $3$  tons,  $X_a = 3$  tons,  $Y_a = 6$  tons,  $AB$   $5$  tons,  $CB$   $3$  tons,  $AC$   $2$  tons.
36.  $BC$   $6.93$ ,  $EF$   $4$ ,  $DE$   $3.46$ ,  $DF$   $4$ ,  $CF$   $4$ ,  $AC$   $4$ ,  $AF$   $8$ .

(In the following trusses, *u. c.* will indicate the members in the upper chord of the truss, reading from the left, *web* will indicate the web members, reading from left, and *l. c.* will indicate the members of the lower chord, reading from the left.)

37. *u. c.*  $3.46, 3.46, 4.62$ , *web*  $0, 2.31$ , *l. c.*  $3.46, 2.31$ .
38.  $P = 204.5$  lbs.,  $R_c = 1521$  lbs.,  $R_a = 834$  lbs.
39.  $BC$   $1195$  lbs.,  $X_a = 1035$  lbs.,  $Y_a = 3403$  lbs.
40.  $X = 0.915$  tons,  $Y_1 = 3.085$  tons,  $Y_2 = 0.085$  tons,  $Y_3 = 1.915$  tons.
41.  $X = 2.381$  tons,  $Y_a = 2.786$  tons,  $Y_c = 7.214$  tons,  
 $Y_b$  (on  $AB$ )  $= 0.786$  tons,  $Y_b'$  (on  $CB$ )  $= 4.214$  tons.
42.  $Y_f = 1562$  lbs.,  $Y_b = 4667$  lbs.,  $Y_d = 2667$  lbs.,  $Y_c = 1105$  lbs.,  
 $X = 0$  at all points.
43.  $P_h = 165$  lbs.,  $P_v = 75$  lbs.      44.  $X = 5W/24$ ,  $Y_a = 2W/3$ ,  $Y_c = W/3$ .
45.  $X = 11W/42$ ,  $Y_a = 3W/4$ ,  $Y_b = 5W/28$ ,  $Y_c = W/4$ .
46.  $X_a = X_b = W/4$ ,  $Y_a = 2W$ ,  $Y_b = W/2$ ,  $CD$   $0.707W$ .

48. In  $CD$  1.044 tons,  $\bar{X}_a = 1.412$  tons,  $Y_a = 2.412$  tons,  
on  $AB$   $X_b = 1.412$  tons,  $Y_b = 0.412$  tons down, on  $BD$   $X_b = 1.112$  tons,  
 $Y_b = 0.588$  tons,  $X_d = 1.112$  tons,  $Y_d = 2.588$  tons.
49. In  $CD$  1.044 tons,  $X_a = 2.235$  tons,  $Y_a = 3.235$  tons;  
on  $BC$   $X_b = X_c = 0.3$  tons,  $Y_b = Y_c = 1$  ton; on  $AB$   $X_b = 2.235$  tons,  
 $Y_b = 1.235$  tons; on  $BD$   $X_b = X_d = 1.935$  tons,  $Y_b = 1.765$  tons,  
 $Y_d = 3.765$  tons.
50.  $Y_n = 2.58$ ,  $Y_m = 3.15$ ,  $X_m = 1$ ,  $R_m = 3.32$ .
51.  $Y_n = 1$  ton,  $Y_m = 3$  tons,  $X_m = 2$  tons,  $R_m = 3.61$  tons.
52. (a)  $R_m = R_n = 1.732$ , (b)  $R_n = 1.5$ ,  $R_m = 2.45$ , (c)  $R_n = 1.80$ ,  $R_m = 2.32$ .
53. 19.8 lbs., 3.26 in. to right. 54. 75.2 lbs.,  $-78^\circ 10'$ ,  $8.37''$ .
55.  $u. c.$  (cf. ans. to Problem 37) 6.93, 5.20,  $\dots$ , web 3.46, 3.46,  $\dots$   $l. c.$  3.46, 6.93.
56.  $u. c.$  3, 2, 2.83; web 3, 1.41, 1, 1.41, 0;  $l. c.$  2, 3, 2.
57.  $u. c.$  1.155, 1.155, 1.732; web 0.577, 0.577;  $l. c.$  5.77, 0.
58.  $u. c.$  2.412, 1.257, 0.782, 4.512; web 0.103, 0.103, 1.052, 1.052;  $l. c.$  3.206, 3.309, 2.256.
59.  $u. c.$  8.08, 5.39, 2.69; web 3, 3.20, 2.69, 2;  $l. c.$  5, 2.5, 2.5.
60.  $u. c.$  18, 8, 2; web 2, 14.14, 6, 8.48, 2;  $l. c.$  8, 2, 2.83.
61.  $u. c.$  1.11, 0.99, 0.91, 1.04; web 0.417, 0.417, 0.217, 0.217;  $l. c.$  1.15, 0.682, 0.899.
62.  $u. c.$  2.5, 1.5, 2.5; web 4, 5, 8;  $l. c.$  1.5, 1.5, 1.5.
63.  $u. c.$  8, 7, 8, 10; web 1.732, 1.732, 3.464, 3.464;  $l. c.$  6.928, 5.196, 8.660.
64.  $u. c.$  0.932W, 0.745W, 0.559W; web 0, 0.186W, 0.083W, 0.236W, 0.333W;  
 $l. c.$  0.833W, 0.833W, 0.667W.
65.  $u. c.$  1.042W', 0.792W', 0.542W', 0.625W'; web 0, 0.417W', 0.186W',  
0.527W', 0.373W', 0, 0, 0, 0;  $l. c.$  1.165W', 1.165W', 0.792W', 0.419W'.
66.  $u. c.$  17.43, 13.69, 9.96, 10.46, 12.69, 14.93; web 0, 4.73, 2.12, 5.99, 6.24, 2.83,  
1, 2.23, 0;  $l. c.$  3.72, 3.72, 7.95, 10.18, 12.17, 12.17.
67.  $u. c.$  0, 0, 1.85, 1.58; web 1, 0, 1, 1.15, 0, 1.15;  $l. c.$  0.412.
68.  $u. c.$  3.064, 2, 2, 3.064; web 1.306, 0, 1.306, 0, 1.306;  $l. c.$  3.10.
69.  $u. c.$  3.06, 2, 3.86, 4.66; web 2.31, 0, 2.31, 0.15, 0, 0.15;  $l. c.$  3.22.
70. (a)  $u. c.$  6.363, 4.242, 4.242, 6.363; web 1.581, 4, 1.581;  $l. c.$  4.743, 4.743.  
(b)  $u. c.$  2, 1, 2, 2; web 2.236, 1.414, 0;  $l. c.$  2.236, 0; (c)  $u. c.$  8.36, 5.242,  
6.242, 8.36; web 3.82, 5.41, 1.58;  $l. c.$  6.98, 4.74.
71. (a)  $u. c.$  4.80, 3.20; web 0, 1.60, 2;  $l. c.$  3.75, (b)  $u. c.$  2.05, 2.05, 1.25, 1.70;  
web 0, 0, 8, 2.05, 0;  $l. c.$  0.58, 0.58, 2.18, 2.18, (c)  $u. c.$  6.85, 5.25, 4.45, 6.50;  
web 0, 1.60, 3.28, 3.65, 0;  $l. c.$  4.16, 4.16, 5.76, 5.76.
72.  $u. c.$  0.417, 0.417, 0.816, 0.816; web 0.5, 0.589, 0.125, 0.394, 0.198, 0.125,  
0.028;  $l. c.$  0, 0.683, 0.833.
73.  $EM$  0.688W,  $MN$  0.325W,  $FN$  0.433W.
74. Stress in horizontal member 13.125 tons.
75. Stresses in members just to left of center 0.839W, 0.417W, 0.5W.
76.  $M = 277.3$  lb.-ft.,  $136^\circ 9'$ ,  $57^\circ 15'$ ,  $115^\circ 38'$
77. 15 lb.-in., 20 lb.-in., 54.54 lb.-in.
78.  $M_y = 200$  lb.-in.,  $M_z = -150$  lb.-in.
79. 51.23 lbs.,  $67^\circ 1'$ ,  $33^\circ 40'$ ,  $60^\circ 48'$ ; 296.8 lb.-in.,  $132^\circ 22'$ ,  $126^\circ 21'$ ,  $122^\circ 37'$ .
80. 50 lbs. along  $OX$ ,  $M_y = 250$  lb.-in.,  $M_z = -400$  lb.-in.

81. 25 lbs. parallel to  $OX$ , through  $(0, 6, 14)$ .  
 82. 375 lbs., 1.663 ft. from  $AB$ , 0.798 ft. from  $BC$ .  
 83. 153.4 lbs.,  $24^\circ 7'$ ,  $68^\circ 59'$ ,  $78^\circ 43'$ ; 1230 lb.-in.,  $123^\circ 34'$ ,  $72^\circ 59'$ ,  $141^\circ 18'$ .  
 85.  $T_a = 39.93$  lbs.,  $T_b = 26.70$  lbs.,  $T_c = 33.37$  lbs.  
 86. 241.9 lbs. through  $(7.64, -9.53, 0)$ ,  $109^\circ 19'$ ,  $34^\circ 13'$ ,  $62^\circ 57'$ ; 628.4 lb.-in.,  $70^\circ 41'$ ,  $145^\circ 47'$ ,  $117^\circ 3'$ .  
 87. 2.632  $T$  in  $CD$ , 3.196  $T$  in  $CE$ , 4.667  $T$  along  $AB$ ,  $V_b = 0.8 T$ ,  $H_b = 0$  in vertical plane.  
 88. 2898 lbs. in  $EF$ , 258.8 lbs. along  $AB$  at  $A$ ,  $X_a = 478.4$  lbs.,  $Y_a = 241.5$  lbs.,  $X_b = 2031$  lbs.,  $Y_b = 241.5$  lbs.  
 89. The force in each leg and in the back-stay is  $W$ .  
 90.  $0.698W$  in back-stay,  $0.783W$  in each leg.  
 91. 164.5 lbs. in  $AD$ , 203.2 lbs. in  $AC$ , 185.7 lbs. in  $FB$ , 219.7 lbs. in  $EB$ .  
 92.  $T = 4550$  lbs.,  $X_a = 864.5$  lbs.,  $Y_a = 1530$  lbs.,  $X_b = 5187$  lbs.,  $Y_b = 3218$  lbs.  
 93.  $0.295W$  in  $AD$ . 94.  $0.551W$  in  $DC$ ,  $0.306W$  in  $DB$ .  
 95. 307 lbs., 307 lbs., 434 lbs. 96. 678 lbs., 678 lbs., 700 lbs.  
 97. 2.598 ft., 1.299 ft. 98. 10.93 ft.  
 99.  $T_1 = 375$  lbs.,  $T_2 = 375$  lbs.,  $T_3 = 250$  lbs.  
 100. (a)  $P_1 = 150$  lbs.,  $P_2 = 80$  lbs.,  $P_3 = 130$  lbs., (b)  $P_1 = 173.7$  lbs.,  $P_2 = 68.17$  lbs.,  $P_3 = 118.2$  lbs.  
 101. 205 lbs., (1.73, 0.293, 1.195). 102. 75 lbs.,  $(-0.467, 0.8, 1.533)$ .  
 103.  $0.6a, 0.375h$ . 104.  $0.6a, 0.375h$ . 114. 2 ft. from center, 1350 lbs., 2550 lbs.  
 115. 3633 lbs., 4967 lbs. 116.  $9r/16$ . 119.  $3.14$  ft.  
 120.  $2a/3$ . 121.  $31.25 bd^2, 2d/3$ . 122. 12938 lbs., (1.57 ft). 123.  $wbd^2/3$ .  
 126. 4418 lbs., 10.056 ft. 127.  $\pi r^2 h w$  lbs.,  $h + r^2/4h$ . 130.  $1.33''$ .  
 131.  $5.32''$ ,  $4.71''$ . 132.  $x = 0.223r$ . 135.  $5.04''$ ,  $4.97''$ ,  $5''$ .  
 136.  $4.94''$ ,  $5.06''$ ,  $5''$ . 137.  $10.11''$ . 138.  $\bar{x}$  of segment =  $7.63''$ ,  $\bar{x}$  of sector =  $5.75''$ . 139.  $\bar{x}$  of segment =  $2(r^2 - d^2)^{3/2}/\text{area}$ ,  $\bar{x}$  of sector =  $2(r)(\text{chord})/3\text{arc}$ . 140.  $3H/56$ . 141.  $1.20''$  from center.  
 142.  $0.6a$ . 143.  $4.52''$ . 144.  $181.4$  sq.-in.,  $10.72''$ . 145.  $0.782a^2$ ,  $0.422a$ . 146.  $943$  cu.-ft.,  $4.66$  ft. 153.  $0.4Ma^2$ . 154.  $M(b^2 + c^2)/5$ .  
 166.  $16.43$  in<sup>4</sup>,  $573.4$  in<sup>4</sup>. 167.  $8.68$  in<sup>4</sup>,  $24.51$  in<sup>4</sup>. 171.  $k_x = 1.12''$ ,  $k_y = 1.88''$ . 172.  $b^2d^2/4$ . 173. (a)  $b^2d^2/8$ , (b)  $b^2d^2/8$ , (c)  $b^2d^2/24$ .  
 174.  $b^3d^3/6(b^2 + d^2)$ . 175.  $(\pi + 2)r^4/16$ . 176.  $22.5^\circ$ ,  $I_{\min.} = 0.253a^4$ ,  $I_{\max.} = 3.081a^4$ . 177.  $\alpha = 27^\circ 17.5'$ ,  $I_{\max.} = 2464$  in<sup>4</sup>,  $I_{\min.} = 255.7$  in<sup>4</sup>. 178.  $\alpha = 121^\circ 37.2'$ ,  $I_{\max.} = 25.37$  in<sup>4</sup>,  $I_{\min.} = 2.85$  in<sup>4</sup>,  $k_{\min.} = 0.737''$ . 179.  $\alpha = 61^\circ 30.9'$ ,  $I_{\max.} = 98.31$  in<sup>4</sup>,  $I_{\min.} = 21.24$  in<sup>4</sup>,  $k_{\min.} = 1.28''$ . 180.  $\alpha = 28^\circ 28'$ .  
 182.  $M(a^2 + b^2)/3$ . 185.  $M(a^2 + b^2)/12$ ,  $M(b^2 + 4a^2)/12$ .  
 186.  $M(a^2 + 12h^2)/20$ ,  $M(4a^2 + 3h^2)/80$ ,  $M(a^2 + 2h^2)/20$ . 187.  $0.4Ma^2$ .  
 188.  $M(a^2 + 12c^2)/24$ . 189.  $M(3b^2 + h^2)/12$ . 194.  $0.346$ ,  $0.271$ ,  $0.297$ .  
 195.  $0.183$ ,  $0.0626$ ,  $0.0626$ . 198. Exact value is  $259.2$  in<sup>4</sup>.  
 201.  $BC$   $0.683W$ ,  $AC$   $0.942W$ . 202.  $x = 3.84$  ft.,  $y = 5.12$  ft., no.  
 203.  $1.278W$ ,  $0.734W$ . 204.  $W_1^2 = W_2^2 + W_3^2$ ,  $W_1^2 > W_2^2 + W_3^2$ ,  $W_1^2 < W_2^2 + W_3^2$ . 205.  $60.2$  lbs.,  $41.9$  lbs.,  $34.5$  lbs.,  $43.2$  lbs.;  $3.76'$ ,  $3.94'$ ,  $2.16'$ ,  $1.35'$ . 206.  $H = 70.95$  lbs., other tensions are  $84.81$  lbs.,  $71.25$  lbs.,  $71.59$  lbs.; lengths are  $2.82'$ ,  $3.59'$ ,  $2.01'$ ,  $1.01'$ .



207. 120.4 lbs., 106.0 lbs., 91.4 lbs., 90 lbs.; 2.68', 3.53', 2.03', 1'.  
 208. 1250 lbs., 750 lbs., 901.5 lbs.; 10', 8', 7.21'. If weights were interchanged equilibrium would be impossible with strings.  
 210. 362 lbs., 317 lbs., 301 lbs., 344 lbs., 475 lbs.; 3.62', 3.17', 2.01', 4.59', 4.75'.  
 211. 1.40, 1.732W. 212. 1.181, 4W. 213. 1 : 1.58 : 0.58.  
 214. Max. tensions are 320 lbs., 500 lbs., 695 lbs. 215.  $H = 15$  lbs.,  $T_{\max.} = 18.03$  lbs., lengths are 24.04', 21.04', 20'. 219. Links, 66.2 lbs., 49.2 lbs., 32.0 lbs., 59.2 lbs.; 6.62', 3.28', 3.20', 3.95'. Strings, 46.1 lbs., 33.5 lbs., 46.1 lbs., 80.8 lbs.; 4.61', 2.24', 4.61', 5.39'.  
 221.  $H = 12500$  lbs., max. thrust = 23600 lbs. 222.  $H = 3333$  lbs., max. thrust = 17200 lbs. 224.  $H = 4000$  lbs.,  $T_1 = 5656$  lbs.,  $T_2 = 7211$  lbs.,  $160y = x^2$ . 225.  $H = 62.5$  lbs.,  $T_{\max.} = 62.7$  lbs.  
 227.  $x_1 = 250/3$ ,  $H = 34722$  lbs.,  $T_{\max.} = 38520$  lbs., length = 257.0 ft.  
 228. 22.5 ft., 130.45 ft. 229. 102.60 ft., 0.54 ft. 231. 0.620".  
 232. 83.165 ft. 233. 20.006 ft. 235. 150.85 ft., 40.535 ft., 90.54w.  
 239. 49.47 ft. 241. 36.53 ft., 238.4 ft., 176.5w lbs.  
 244.  $T_{\max.} = 11.41$  lbs.,  $s = 205.3$  ft.,  $e = 0.074''$ . 245.  $H/w = 148.5$ , span 59.61 ft. 246. 8.734 ft. 248. 1.073".  
 249.  $s$  is positive and increasing for  $0 < t < 3$  sec.,  $s$  is positive and decreasing for  $3 < t < 6.10$  sec.,  $s$  is negative for  $t > 6.10$  sec.,  $v$  changes sign at  $t = 3$  sec. 250.  $v = 64 - 20t$ ,  $s = 30 + 64t - 10t^2$ .  
 251.  $s = 2 \cos 3t$ ,  $v = -6 \sin 3t$ ,  $a = -18 \cos 3t$ ,  $\alpha = -9s$ .  
 252.  $v = -6 \sin 2t$ ,  $s = 3 \cos 2t$ . 253.  $v = (20/s - 1/50)^{1/2}$ .  
 255. Max. distance from  $O = v_0/\sqrt{k}$ , oscillates between  $-v_0/\sqrt{k}$  and  $v_0/\sqrt{k}$ .  
 256.  $v_0^2/2g$ . 265. 29.24 lbs. 266.  $A$  2.8 ft.,  $B$  7 ft.,  $C$  4.9 ft.,  $T = 17.39$  lbs.  
 271. 1 : 5, 3 : 10, between 0.2 and 0.3. 273. 24.49 ft./sec., 21.45 ft., 40.32 ft./sec., 25.32 ft./sec., 840 lbs. 274. 11.13 ft./sec., 40 ft./sec.  
 275. Max.  $v = 8.38$  ft./sec.,  $-0.634$  ft., 2.59 ft./sec., 7.26 ft./sec.  
 276. 0.505 sec., 3.108 ft./sec. 284. 2.5 lbs., 2.24 lbs., 6.57 lbs.  
 285. 9.23 lbs. 286. 119.9 lbs. 288. At  $60^\circ$  13.90 ft./sec.,  $-2.5$  lbs.; at  $90^\circ$  19.66 ft./sec.,  $-10$  lbs.; at  $150^\circ$  24.07 ft./sec.,  $-17.5$  lbs.; at  $180^\circ$  27.80 ft./sec.,  $-25$  lbs. Change occurs at  $48^\circ 11'$  from top.  
 289. 129.9 lbs., 150 lbs. 290. 6.19 ft.  
 292. Down plane  $R_{\max.} = v_0^2/[g(1 - \sin \beta)]$ , when  $\alpha = 45^\circ - \beta/2$ .  
 293. 98.8 ft./sec., 303 ft. 294. 0.48 in., 4.03 ft. 296.  $34^\circ 50'$ , 540 ft.  
 299. 123.1 ft./sec. 300. 25.38 ft./sec., 47.48 ft./sec. 301. 14.77 ft./sec., 13.90 ft./sec., 13.90 ft./sec. 302.  $\sqrt{gr}$ ,  $\sqrt{5gr}$ ,  $6W$ .  
 303. 171.2 ft./sec., 139.0 ft./sec.,  $35^\circ 44'$ . 304. 39.15 in.  
 305. 1.933 sec., 1.918 sec. 306. 1.0077 sec. 307. 0.554 sec.  
 308. 9.893. 309. 32.19 ft./sec<sup>2</sup>. 311.  $x = a(\theta + \sin \theta)$ ,  $y = a(1 - \cos \theta)$ .  
 312. 2.45 in., 9.79 in., 38.2 lbs., 9.81 lbs. 313. 142.6 r.p.m. 2.31 lbs., 4.62 lbs.  
 314.  $v^2 = gr \tan \theta$ ,  $7^\circ 37'$ . 315.  $v^2(b^2 - d^2)^{1/2} = grd$ . 316. 13.99 mi.,  $W$   $30^\circ 22' S$ . 317. 64.78 mi./hr.,  $W$   $25^\circ 53' S$ . 318. 25.40 ft./sec.,  $55^\circ 17'$ . 319. 78.59 mi./hr., 102.0 mi./hr. 320. 1 hr. 2.9 min.  
 321. Equal speeds. 323.  $2v_w$ . 324. 6.49 mi./hr., 49.39 mi./hr.  
 326.  $\omega_{\max.} = 4\pi^2/3$ ,  $\alpha_{\max.} = 16\pi^3/3$ . 328. 40 sec. 333. 10 ft./sec., 14.84 ft./sec.; 0 ft./sec., 12.5 ft./sec.; 8.66 ft./sec., 17.05 ft./sec.



334. velocity =  $(v/r)(R^2 + r^2 + 2Rr \sin \theta)^{1/2}$ ,  
 accel. =  $(1/r^2)(R^2v^4 + 2aR^2r^2 \cos \theta v^2 + R^2a^2r^2 + 2Rr^3a^2 \sin \theta + a^2r^4)^{1/2}$ .
339. 9.55 in. below  $C$ ,  $v_a = 13.51$  ft./sec. to left,  $v_b = 53.51$  ft./sec.,  
 $v_d = 39.02$  ft./sec.
341.  $\theta = 0$   $v' = 0$ ,  $\omega' = 1.257$  rad./sec.;  $\theta = 30^\circ$   $v' = 0.908$  ft./sec.,  
 $\omega' = 1.094$  rad./sec.;  $\theta = 90^\circ$   $v' = 6.283$  ft./sec.,  $\omega' = 0$ ;  $\theta = 180^\circ$   $v' = 0$ ,  
 $\omega' = -1.257$  rad./sec. 344.  $\omega' = 1.094$  rad./sec.,  
 $\alpha' = 3.847$  rad./sec<sup>2</sup>. clockwise,  $a' = 38.22$  ft./sec<sup>2</sup>.
349.  $\omega' = -0.529$  rad./sec.,  $\alpha' = 5.78$  rad./sec<sup>2</sup>,  $a' = -22.41$  ft./sec<sup>2</sup>.
350.  $\omega' = 1.37$  rad./sec.,  $\alpha = 18.18$  rad./sec<sup>2</sup>,  
 $\alpha' = 34.93$  rad./sec<sup>2</sup>. counter-clockwise.
351. 50 r.p.m. about a line through  $O$  in the plane  $AOB$  at  $36^\circ 52'$  with  $OB$ .
359. 32.12 ft./sec. 361.  $3.52''$ ,  $46.4''$ . 362. 65.52 ft./sec.,  $2.83''$ .
363.  $Ws_0/2$  in.-lbs.,  $3Ws_0/2$  in.-lbs. 364. 0.31 in., 15500 lbs.;  $0.44$  in.,  
 11000 lbs. 366.  $6.16''$ . 371. 8.29 ft. 374. 15.01 ft./sec.,  
 6.60 lbs. tension; 18.85 ft./sec., 13.24 lbs. compression.
375.  $v$  of  $W = 4.97$  ft./sec.,  $v$  of  $W_1 = 7.46$  ft./sec.,  $\omega = 3.73$  rad./sec.,  
 $T = 49.04$  lbs. 377. 15.1, 35.5. 380. 94.48. 387. (a) 87.9 lbs.,  
 (b) 67.33 lbs., (c) 66.16 lbs., least  $P = 66.02$  lbs., at  $41^\circ 19'$  with horizontal.
388. (a) 33.82 lbs., (b) 32.67 lbs., (c) 35.74 lbs.,  
 least  $P = 32.03$  lbs. at  $18^\circ 41'$  with horizontal. 392. 818.3 lbs.
400. 1.43. 401. 197.6 lbs. 402. 641 lbs., 577.4 lbs. 403. 523 lbs.
405. 35.84 ft. 408. 46.12 lbs. 409. 405.2 lbs. 411. 402.5 lbs.
412. 114.7 lbs. per inch of width. 413.  $7.18''$ . 416. 9.58.
417. 23.8 hp., 91 ft./sec. 418. Center at  $x = 0$ .
420.  $U = km_1m_2/(x^2 + y^2 + z^2)^{1/2}$ . 421. (a)  $N_1 = 278.1$  lbs.,  $N_2 = 221.9$  lbs.;  
 (b)  $N_1 = 315.6$  lbs.,  $N_2 = 184.4$  lbs.,  $h = 0.25$  ft.
422.  $a = 2.415$  ft./sec<sup>2</sup>,  $N_1$  (left) = 62.29 lbs.,  $N_2 = 37.71$  lbs.
424.  $a = 2rg/h$ . 426.  $2Mr\omega^2/3\pi$ . 427. At  $60^\circ H = 0.974W$ ,  $V = 1.938W$ ;  
 at  $90^\circ H = 0$ ,  $V = 2.5W$ . 429.  $H = 3Wa^2 \sin \theta (3 \cos \theta - 2)/(4a^2 + b^2)$ ,  
 $V = W[1 - 3a^2(1 + 2 \cos \theta - 3 \cos^2 \theta)/(4a^2 + b^2)]$ .
434. At upper bearing  $X = -139.7$  lbs.,  $Z = -5.65$  lbs.,  
 at lower bearing  $X = -54.1$  lbs.,  $Z = -2.83$  lbs.,  $Y = 120$  lbs.
435.  $\theta = 90^\circ$   $V_a = 182$  lbs.,  $V_b = 339$  lbs.,  $H_a = -463$  lbs.,  $H_b = -497$  lbs.;  
 $\theta = 150^\circ$   $V_a = 1079$  lbs.,  $V_b = 692$  lbs.,  $H_a = -395$  lbs.,  $H_b = 13$  lbs.
437.  $a = 4.663$  ft./sec<sup>2</sup>. down plane,  $F = 20.13$  lbs. down plane,  $N = 182.8$  lbs.,  
 $s = 77.0$  ft. 438.  $Q = 535$  lbs.,  $N = 3385$  lbs.,  $T = 2756$  lbs.
440. Centers are  $A(0.4d, -0.4d)$ ,  $B(-0.4d, 0.4d)$ .
443. At end  $A$   $x = 11.71''$ ,  $y = 1.56''$ ; at end  $B$   $x = 12.14''$ ,  $y = 20.32''$ .
448. 0.0191 slug ft<sup>2</sup>. 456. 2611 ft./sec., 99.68%, 30.34 lbs.
458. (a) 0.51 ft./sec., (b) 6.836 rad./sec., (c) The rod swings completely around,  
 (d) 49.36%. 459. (a) 2.60 ft./sec., (b) 2.74 ft./sec., (c) 3.43 rad./sec.,  
 (d) 82.19%. 461. 36.62 ft./sec. at  $129^\circ 26'$  with pos.  $x$  axis,  
 and 29.65 ft./sec. at  $-30^\circ 24'$  with pos.  $x$  axis.
462.  $v' = v(1 + e^2 \cos^2 \alpha)^{1/2}$ ,  $\tan \beta = \tan \alpha/e$ . 463.  $v = 41.96$  ft./sec.,  
 $e = 0.704$ . 464.  $9''$  from the other end. 465.  $l/3$  from the free end.
466. 1 ft. from an end. 467.  $h/3 - r^2/(2h)$  from the other end; no.

468.  $h_x = 78.92$ ,  $h_y = 0$ ,  $h_z = 0$ ,  $h_x' = 78.92$ ,  $h_y' = -78.05$ ,  $h_z' = 0$ ,  
 $h_0 = 78.92$ ,  $h_0' = 1110$ .      473.  $P' = 30.21$ ,  $P = 9.79$ .  
474. 39.15 r.p.m.      475. 52.2 r.p.m.      477. 59.72 ft./sec., 71.78 ft./sec.  
479. 13010 lbs. upward and 11510 lbs. downward.  
482.  $\omega'$  from 0 to 2.733 rad./sec.,  $\theta$  from  $30^\circ$  to  $30^\circ 55'$ .

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